Brane Partons and Singleton Strings

J. Engquist[1] and P. Sundell[2]

Department of Theoretical Physics
Uppsala University
Box 803, SE-751 08 Uppsala, Sweden

Abstract

We examine p-branes in AdS_D in two limits where they exhibit partonic behavior: rotating branes with energy concentrated to cusp-like solitons; tensionless branes with energy distributed over singletonic partons on the Dirac hypercone. Evidence for a smooth transition from cusps to partons is found. First, each cusp yields $D - 2$ normal-coordinate bound states with protected frequencies (for $p > 2$ there are additional bound states); and can moreover be related to a short open p-brane whose tension diverges at the AdS boundary leading to a decoupled singular CFT at the "brane at the end-of-the-universe". Second, discretizing the closed p-brane and keeping the number N of discrete partons finite yields an $\mathfrak{sp}(2N)$-gauged phase-space sigma model giving rise to symmetrized N-tupletons of the minimal higher-spin algebra $\mathfrak{ho}_0(D-1,2) \supset \mathfrak{so}(D-1,2)$. The continuum limit leads to a 2d chiral $\mathfrak{sp}(2)$-gauged sigma model which is critical in $D = 7$; equivalent *á la* Bars-Vasiliev to an $\mathfrak{su}(2)$-gauged spinor string; and furthermore dual to a WZW model in turn containing a topological $\widehat{\mathfrak{so}}(6,2)_{-2}/(\widehat{\mathfrak{so}}(6) \oplus \widehat{\mathfrak{so}}(2))_{-2}$ coset model with a chiral ring generated by singleton-valued weight-0 spin fields. Moreover, the two-parton truncation can be linked via a reformulation *á la* Cattaneo-Felder-Kontsevich to a topological open string on the phase space of the D-dimensional Dirac hypercone. We present evidence that a suitable deformation of the open string leads to the Vasiliev equations based on vector oscillators and weak $\mathfrak{sp}(2)$-projection. Geometrically, the bi-locality reflects broken boundary-singleton worldlines, while Vasiliev's intertwiner κ can be seen to relate T and R-ordered deformations of the boundary and the bulk of the worldsheet, respectively.

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[1] E-mail: `johan.engquist@teorfys.uu.se`
[2] E-mail: `per.sundell@teorfys.uu.se`

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1 INTRODUCTION AND SUMMARY

1.1 General Discussion

The most intriguing property of String Theory is the absence of a global minimum principle. This is essentially a manifestation of general covariance. The main obstacle in formulating the theory covariantly is the fact that the standard quantization takes place inside spacetime, whereas the covariant formulation – whether perturbative or not – ought to involve a dynamical reconstruction of spacetime.

This motivates revisiting the principles of gauging in field theory, focusing on stringy forms of covariance, with the aim of developing tools for extracting background independent physical information. Indeed, such a program based on higher-spin symmetries – initiated and pursued in its early days mainly by Fradkin and Vasiliev in a fashion quite independent from parallel trends in String Theory – led early to the Vasiliev equations in $D = 3$ and $D = 4$ [1, 2, 3], whose higher-dimensional generalizations have started to be understood more recently [4, 5, 6, 7, 8, 9, 10, 11, 12].

A key development has been the unfolding principle [13, 14, 15, 16], according to which the geometry is determined by a in general deformed algebra of differential forms arising from gauging. In this spirit, here we examine a "doubling" proposal formulated on a fiberbundle $E[\mathcal{M}, \mathcal{Z}]$ (for recent, related work on geometric quantization see [17]), whereby a quantum string living in a rigid phase space $\mathcal{Z}$ produces a zero-form master field – the phase-space analog of the string field – in turn determining a one-form master field – that contains a vielbein as well as higher-spin gauge fields – describing a classical spacetime $\mathcal{M}$ through the unfolding principle.

This also reflects the spirit of open-closed string duality [18] (see [19, 20] for recent developments), with the open-string side corresponding to the fiber theory and the closed-string side to the generally covariant classical master-field equation on the base-manifold $\mathcal{M}$ resulting from projection of the quantum master-field equation on E. We emphasize, however, that the fiber theory is ultimately a topological closed string/open membrane.

The resulting master-field equations harmonize with the strong duality principle that String Field Theory, or M-theory, is completely void of free parameters. Viewed from this angle, the Landscape [21] would unfold via expansions of a strictly non-perturbative Master Equation around solutions exhibiting various unbroken symmetries in turn gauged via current-photon couplings and subsumed in a "kitchen-sink" fashion into a truly unified symmetry algebra.

Here we emphasize a crucial – though not completely uncontroversial – standpoint, namely that the string tension is a background *dependent* quantity determined by the solution, whereby tensionless limits [22, 23, 24, 25] open windows into the Landscape which may ultimately turn out to be wider than those offered by traditional tensile strings and supergravity. Moreover, since excitations of strings and other extended objects carry general angular momenta, the unifying algebra must incorporate higher-spin symmetries [1, 26, 27, 28, 4, 6, 7, 29] (see also [30]). The developments initiated by [31] later led to the appreciation that these symmetries play a natural role within holographic space-time reconstruction [32, 28, 33, 4, 6, 35, 34, 36, 37, 38, 39, 40, 41].

This paper treats aspects of tensionless limits, and how the resulting brane dynamics is described by phase-space strings subsuming higher-spin gauge theories. Our main working hypothesis is that the standard target space is actually part of a rigid fiber, whose unbroken structure group emerges in the tensionless limit, whereupon the unfolding yields the true spacetime, where, for example, references can be made to holographic duals.

1.2 Higher-Spin Gauge Theory and Singleton Strings

The higher-spin gauge theories are generalizations of pure AdS gravity, in which the metric is accompanied by an infinite tower of higher-spin fields *and* special sets of lower-spin fields. In the minimal setting, the spectrum consists of massless symmetric Lorentz tensors of rank $s = 0, 2, 4, \ldots$ making up an irreducible massless higher-spin multiplet, which can be "packed away" into a convenient master field introducing an oscillator in turn playing a crucial role for the formulation of the full equation.

The theories are thus based on associative oscillator algebras intimately related to singletons – the remarkable ultra-short unitary $\mathfrak{so}(D-1, 2)$ irreps without flat-space limit – and come in varieties equipped with additional matrix groups. This points to a geometrical realization of singleton-valued Chan-Paton factors via open topological strings[1].

In Section 4 we shall see that the D-dimensional Vasiliev equations with vector oscillators and weak $\mathfrak{sp}(2)$ projection [9, 29] originate from a topological open string on the phase

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[1]The idea of realizing ordinary finite-dimensional Chan-Paton factors as a two-dimensional topological theory of fermionic "dipoles" dates back to [42] and has more recently been examined in using the superembedding approach to non-abelian Born-Infeld theory [43].

space Γ of Dirac's D-dimensional hypercone [44], described by the sigma-model action

$$S \;=\; \frac{1}{2}\int_{\Sigma}\left(DY^{Ai}\wedge\eta_{Ai} + \frac{1}{2}\eta^{Ai}\wedge\eta_{Ai} + \xi^{ij}F_{ij}\right)\;,\tag{1.1}$$

where Y^{Ai} parameterize a $2(D+1)$-dimensional ambient phase space $\mathcal{Z}$. The classical model has a critical point with unbroken $\mathfrak{sp}(2)$ gauge symmetry, where the zero-modes (x^A, p^A) of $Y^{iA}|_{\partial\Sigma}$ are confined to Γ, *viz.*

$$x^A x_A \;=\; x^A p_A \;=\; p^A p_A \;=\; 0\;.\tag{1.2}$$

In a slight abuse of terminology, this implies that $\partial\Sigma$ carries non-compact Chan-Paton factors valued in the scalar singleton and anti-singleton (see Appendix A for conventions)

$$\mathfrak{D} \;\equiv\; \mathfrak{D}(\epsilon_0, (0))\;,\qquad \widetilde{\mathfrak{D}} \;\equiv\; \widetilde{\mathfrak{D}}(-\epsilon_0, (0))\;,\qquad \epsilon_0 \;=\; \frac{D-3}{2}\;.\tag{1.3}$$

Here we stress that, unlike the introduction of ordinary compact Chan-Paton factors – which are typically introduced by hand – the singleton-valued factors arise geometrically from the topological action (1.1).

Treating the singleton as a point-particle [45, 46, 47], the phase-space formulation concerns deformations generated by insertions of vertex operators on closed singleton worldlines in turn representing traces (see Section 3.6). The vertices are functions on $\mathcal{Z}$ subject to $\mathfrak{sp}(2)$ gauge conditions [48] projecting them to operators mapping $\widetilde{\mathfrak{D}}$ to $\mathfrak{D}$, *i.e.* elements of $\mathfrak{D}\otimes\widetilde{\mathfrak{D}}^\star$, representing external *massless two-singleton composites*. As we shall see, these states are given by the Flato-Fronsdal formula [49, 29]

$$\mathfrak{D}\otimes\mathfrak{D} \;=\; \bigoplus_{s=0}^{\infty}\mathfrak{D}(2\epsilon_0 + s, (s))\;.\tag{1.4}$$

This germ of an extended object dates back to the highly influential work of Flato and Fronsdal [49, 50, 51] who introduced the notion of *local and bi-local master fields*, *i.e.* functions on $\mathcal{Z}$ and $\mathcal{Z}\times\mathcal{Z}$, and was later turned by Fradkin and Vasiliev [1, 2] into an elegant algebraic machinery used to write the full higher-spin equations [2], while the geometric aspects have been pursued more recently mainly by Bars et al [52, 53, 48].

Here we unify the approaches using Cattaneo-Felder-Kontsevich's stringy reformulation of phase-space quantum mechanics [54, 55] – leading to (1.1) – refined further by an algebraic treatment of embedding-field *branch points where the boundary-singleton worldline breaks to form an asymptotic two-singleton composite in turn described by a bi-local operator reducing to a local operator only at the linearized level.* This provides a natural geometric realization of Vasiliev's algebraic structures and indeed facilitates the construction of a deformation potentially giving rise to the full Vasiliev equation.

To specify our results, let us first recall the D-dimensional Vasiliev equations [9, 11]. These can be written in a *strongly $\mathfrak{sp}(2)$-projected* form as [11]

$$\widehat{d}\,\widehat{A} + \widehat{A}\star\widehat{A} + \widehat{\Phi}\star\widehat{J}' \;=\; 0\;,\qquad \widehat{d}\,\widehat{\Phi} + [\widehat{A},\widehat{\Phi}]_\pi \;=\; 0\;,\tag{1.5}$$

$$\widehat{K}_{ij}\star\widehat{\Phi} \;=\; 0\;,\tag{1.6}$$

where $\widehat{d} = d + d'$, with d and d' given by the exterior derivatives on a *commutative space-time manifold* $\mathcal{M}$ – which we shall refer to as the *unfold* – and a *non-commutative phase space* $\mathcal{Z}$, respectively; $\widehat{A}$ and $\widehat{\Phi}$ are *adjoint and strongly projected twisted-adjoint bi-local master fields* of total degree 1 and 0, respectively, where the total degree is the sum of form degrees on $\mathcal{M}$ and $\mathcal{Z}$; the master-field components are functions of $(x^\mu, z_i^A) \in \mathcal{M} \times \mathcal{Z}$ taking values in a fiber spanned by functions of $y_i^A \in \mathcal{Z}$; $\widehat{J}'$ is a fixed *intertwiner* of degree $(0, 2)$, and $[\cdot, \cdot]_\pi$ is the twisted-adjoint representation map; and, $\widehat{K}_{ij}$ are dressed-up versions of the $\mathfrak{sp}(2)$ generators in (1.2).

Originally, the equations were presented in the *weakly $\mathfrak{sp}(2)$-projected form* [9]

$$(\widehat{F} + \widehat{\Phi} \star \widehat{J}') \star \widehat{M} \ = \ 0 \ , \qquad \widehat{D}\widehat{\Phi} \star \widehat{M} \ = \ 0 \ , \tag{1.7}$$

where the field strength and covariant derivative are the same as in (1.5), and $\widehat{M}$ is a dressed-up $\mathfrak{sp}(2)$-projector – or phase-space propagator – obeying

$$\widehat{K}_{ij} \star \widehat{M} \ = \ 0 \ . \tag{1.8}$$

The weak projection induces shift symmetries that eliminate the Lorentz traces in the fiber indices of the zero-form *and* the one-form, resulting in a non-linear system built on Fronsdal's doubly traceless free-field equations [56]. The strong projection, on the other hand, only reduces the zero-form letting the gauge fields adjust to the source and leading [57, 58, 11] to a system built on Francia-Sagnotti's geometric compensator form of the free-field equations [59, 60], containing $\mathfrak{sl}(D)$-tensor gauge fields closely connected to the naive tensionless limit of flat-space string field theory. The precise relation between the two types of projections remains to be uncovered, and the details of the topological open string on the Dirac hypercone may provide useful clues to this end.

Three salient features of the Vasiliev equations are:

1. *Referring $\mathcal{M}$ to a D-dimensional spacetime (see Section 4.7), the equations yield ghost and tachyon-free generally covariant field equations*, with a well-defined weak-field expansion, and a strongly coupled yet controllable derivative expansion, with fundamental length scale equal to the radius of an unbroken anti-de Sitter vacuum.

2. The equations are *unfolded, i.e. written in terms of differential forms only without any explicit contractions of curved indices using the metric*, in turn inducing manifest invariance under *homotopy transformations of $\mathcal{M}$ preserving the cohomological data contained in the master fields* [61, 62].

3. The complete perturbative spectrum is contained in the twisted-adjoint initial condition $\widehat{\Phi}(x, z; y)|_{x=z=0}$.

The equations exhibit nonetheless a number of tantalizing properties: first, the structure of the bi-local $\star$-product algebra, *viz.*

$$\widehat{f}(y, z) \star \widehat{g}(y, z) \ = \ \int \frac{d^{2(D+1)}\eta \, d^{2(D+1)}\xi}{(2\pi)^{2(D+1)}} e^{i\eta^{iA}\xi_{iA}} \widehat{f}(y + \xi, z + i\xi) \widehat{g}(y + \eta, z - i\eta) \ , \tag{1.9}$$

such that y_i^A and z_i^A, which commute to each other, have the *"skew" mutual contractions*

$$y_i^A \star z_j^B - y_i^A z_j^B \;=\; i\epsilon_{ij}\eta^{AB}\,, \qquad z_i^A \star y_j^B - z_i^A y_j^B \;=\; -i\epsilon_{ij}\eta^{AB}\,. \qquad (1.10)$$

Second, the detailed structure of the *intertwiner, viz.*

$$\widehat{J}' \;=\; v_A v_B dz^{iA} \wedge dz_i^B \kappa\,, \qquad \kappa \;=\; \exp\left(v_A v_B y^{Ai} z_i^B\right)\,, \qquad v^A v_A \;=\; -1\,, \qquad (1.11)$$

serves a dual purpose, projecting also the linearized zero-form $\Phi(x;y)$ to $\Phi(x;y^{ai},y^i)\star \kappa|_{z=0} = \Phi(x;y^{ai},0)$ that contains generalized Weyl tensors sourcing the spin $s=1,2,3,\ldots$ curvatures on $\mathcal{M}$ upon unfolding. Third, shrinking $\mathcal{M}$ to a point, denoted here by priming, leads to *open-string-field-like equations on $\mathcal{Z}$ with a consistent "classical anomaly", viz.*

$$(d'\widehat{A}' + \widehat{A}' \star \widehat{A}' + \widehat{\Phi}' \star \widehat{J}') \star \widehat{M}' \;=\; 0\,, \qquad (1.12)$$

$$(d'\widehat{\Phi}' + [\widehat{A}', \widehat{\Phi}']_\pi) \star \widehat{M}' \;=\; 0\,, \qquad (1.13)$$

with the BRST-like exterior derivative

$$d' \;=\; dz^{Ai} \frac{\partial}{\partial z^{Ai}}\,. \qquad (1.14)$$

In the geometric realization, the two end-points of the string will be coordinatized by y_i^A and z_i^A. The topological gauge symmetries allow observables to depend on the center-of-mass but not the relative distance between the end-points, which is equivalent to taking the linearized BRST-operator to be given by (1.14) with dz_i^A identified as one of the non-zero-mode oscillators of the shift-symmetry C^{Ai}-ghost. Moreover, the topological Green functions, which are essentially locally constant phase factors (see Section 4.4), contain long-range correlations between y_i^A and z_i^A leading to (1.10). Finally, we propose that the intertwiner κ arises in the map taking an operator at $\partial\Sigma$ representing an initial state of the radial-ordered evolution on Σ to a corresponding operator inserted into the path-ordered evolution along $\partial\Sigma$ (see Section 4.3).

We are led to propose in an *ad hoc* fashion (see Section 4.5) that *the internal part of the weakly projected Vasiliev equations follows from demanding exact marginality of the phase-space observables*

$$\widehat{Tr}_\pm\left[T\left(\exp \oint_{\partial\Sigma} \widehat{A}'\right) R\left(\exp i \iint_\Sigma v_A v_B dY^{iA} \wedge dY_i^B \widehat{\Phi}'\right)\right]\,, \qquad (1.15)$$

where the master fields are integrated in one of their arguments, identified on $\partial\Sigma$ with z_i^A, while the other argument is attached to a fixed base-point on $\partial\Sigma$, identified with y_i^A. Thus, the consistent classical anomaly in (1.12) is an "inflow" from the bulk, while the non-abelian structures arise from a Wilson-loop on the boundary. Moreover, the fact that the Vasiliev equations are built from a one-form and a zero-form but no higher-rank forms is the result of the world-sheet geometry and that a Lorentz invariant Wess-Zumino potential can be built from d^2Y and the zero-form. In this spirit, it is natural to expect that the original 4D spinor-oscillator Vasiliev equations originate from a topological open spinor string, as we shall discuss in Section 4.6.

Our arriving at (1.12) and (1.13) starting from (1.15) relies on a number of assumptions: *a)* that the classically consistent truncation of the $\mathfrak{sp}(2)$-triplet sector in (1.1) – which results

in a free world-sheet theory subject to a subsidiary $\mathfrak{sp}(2)$ constraint – can be implemented at the quantum level by means of an insertion of $\widehat{M}$ into the free-field trace; *b)* that the observables can be constructed entirely in terms of the embedding field Y_i^A; *c)* that it is consistent to drop terms on $\partial\Sigma$ that are exact with respect to the shift-symmetry BRST operator – which contain non-zero-mode oscillators.

Clearly, assumption *(a)* incorporates the weakly projected formalism from the outset, while the strongly projected equation relates more naturally to correlators with un-amputated external legs. Another subtlety resides in the implementation at the full level of the $\mathfrak{sp}(2)$-symmetry of the linearized theory. A rigorous treatment of the $\mathfrak{sp}(2)$-gauging may furthermore result in a critical dimension. On the one hand, this would be unexpected, since the model is a reformulation of a point-particle, but on the other hand the observable involves a genuinely two-dimensional surface term.

Due to the close analogy with ordinary open-string field theory, we expect assumption *(b)* to be valid at the classical level, while the quantization of the open string will require a thorough understanding of moduli associated with shift-symmetry ghosts. The truncation *(c)* is ideally also consistent and simply amounts to the dressing of the $\mathfrak{sp}(2)$-projector described above, as otherwise there would arise a puzzle, in that the oscillator corrections are not suppressed by the analog of α' due to the topological nature of (1.1). One possibility is, of course, that there are many different separately consistent but mutually inconsistent master field equations that can be built on top of the linearized content of the open string.

Conceptually speaking, the possibility to define Higher-Spin Gauge Theory as the "Theory of Phase-Space Strings" may provide a powerful constructive principle. At this moment, however, we are forced to leave this issue open, but we hope eventually to gain such an understanding of the microscopic nature of the open string interactions that we can derive (1.15) and its consequences from first principles, *i.e.* an open string vertex and BRST operator. Nonetheless, we think it is clear, and we want to emphasize, that *the quantization takes place in phase space while the chronologically ordered presentation of the physical information in the string field is given in the space-time unfold* $\mathcal{M}$, and we shall touch this issue in somewhat more detail in Section 4.7. A list of challenging problems include: *i)* the formulation of quantum consistent phase-space and spinor strings, involving full supersymmetric higher-spin gauge theories in $D > 4$ (where presently only partial results exist [6, 8, 37, 29]) and incorporation of massive states via topological chiral closed strings/open membranes (which we shall touch below); *ii)* the construction and examination of classical solutions to the master field equations [63, 64, 65], requiring deformed oscillators [63, 65], charges and on-shell actions [16]; *iii)* various aspects of the doubling approach, such as the manifestly higher-spin covariant formulation of the unfolded geometry, the relation between unfolded and Lagrangian quantum corrections, and holographic aspects.

Our proposal for relating the Vasiliev equation to singleton deformation quantization, relies on open-string tree-level amplitudes. At this level, the deformation quantization requires a well-defined notion of hermiticity of the master fields, while it does not require unitarity of the underlying singleton representation. The formalism should therefore apply equally well to general space-time signatures. Indeed, the vector-oscillator form of the Vasiliev equations exist in spacetimes with more general signatures [9, 11].

Ultimately, in analogy with ordinary supergravity and open-string theory, the quantum

theory should become part of a fuller theory with massless two-singleton as well as massive multi-singleton states – admitting the Vasiliev equations as a classically consistent truncation. Indeed, the resemblance between the spectrum of massless fields (1.4) and the tensionless limit of the leading flat-space Regge trajectory [23, 66, 60, 67] can be made into a more precise correspondence, which includes massive multi-singletons and higher trajectories, by using supersymmetry and holography arguments [68, 69, 37, 40, 70]. Clearly, this motivates establishing a more direct link between the phase-space approach and the tensionless limit of (bosonic) p-branes in anti-de Sitter spacetime.

1.3 TENSIONLESS LIMITS

To follow extended objects to small tension, it is useful to picture the brane phase space covariantly as the space of all classical solutions. As the tension of a classical rotating closed brane is switched off adiabatically, the centrifugal force causes the energy density to accumulate at ultra-relativistic *cusps* connected to the center-of-mass region by thinly stretched portions of the brane [71, 72]. As we shall see, this naive argument fails in flat spacetime, while it holds in anti-de Sitter spacetime, where the background curvature exerts a tidal force acting together with the centrifugal force to induce an enhanced accumulation of energy-momentum to the cusps. This leads to a potential well in the normal-coordinate mass term, as we shall describe in more detail below. In the case of strings, the well contains $D-2$ bound states, giving rise to additional bound-state oscillators in the normal-coordinate field theory, out of which $D-3$ have protected frequencies and the remaining one has a small fixed anomalous dimension. Remarkably, this result extends to membranes, while the potential well contains additional bound states for $p > 2$.

Thus, the negative cosmological constant is forced if one requires that rotating branes configurations fill a distinct *partonic region of phase space*, parameterized by

$$\bigcup_{N=2}^{\infty} \{X^m(\tau;\xi)\}_{\xi=1}^{N} , \tag{1.16}$$

where N runs over the number of partons and $X^m(\tau;\xi)$ denote their space-time trajectories. In the "dilute gas" approximation, this region is closed under time-evolution *i.e.* the total Hamiltonian

$$H \simeq \sum_{N=2}^{\infty} H_N(\{X^m(\tau;\xi)\}) , \tag{1.17}$$

where the omitted subleading interactions are off-diagonal elements suppressed by inverse powers of semi-classical parameters followed by multi-body interactions suppressed by powers of the space-time Planck constant.

Focusing on the leading part, any amount of tension – no matter how small it is – leads to attraction between the cusps keeping them on large circular orbits. Thus, in addition to point-particle-like mass-shell conditions, the cusps obey additional Gauss'-law-like constraints, leaving $D-2$ physical normal-coordinate oscillators, since a cusp at the end of a long stretched portion of a p-brane is free to move in $D-2$ overall transverse directions.

The results referred to above point to a *transition from cusps to singletons as the tension is switched off*. There are several ways of understanding this: first, if the tension is

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switched off adiabatically, then each cusp becomes a partonic lump following a massless geodesic confined to a $(D-2)$-dimensional hypersurface, identifiable as a short open p-brane attached to a $(D-2)$-brane in a decoupling limit in which the effective p-brane tension is sent to infinity resulting in massless quanta on the $(D-2)$-brane at the end-of-the-universe [26, 73, 74]. Lending the terminology of [75, 76, 77], the $(D-2)$-brane solution is a *giant vacuum* with a *singular conformal field theory* [78] living on it. The resulting *tensionless spectrum of space-time one-particle states consists of symmetrized multipletons*[2]

$$\mathcal{S} \; = \; \sum_{N=1}^{\infty} [\mathfrak{D}^{\otimes N}]_{\text{symm}} \; , \tag{1.18}$$

in agreement with the above interpretation and the fact that the normal-coordinate realization of the soliton gas obeys Bose symmetry in the leading order of the semi-classical expansion. In the tensionless limit, mixed multipletons should arise as space-time multiparticle states. These are shifted into the spectrum of one-particle states by tensile deformations, such as those making up the non-trivial part of stringy spin-chains [79]. We also note that additional bound states, do not arise in the maximally supersymmetric cases, where Type IIB open strings end on $D3$-branes at the boundary of $AdS_5 \times S^5$, and $M2$-branes end on $M2$ or $M5$-branes at the boundary of $AdS_4 \times S^7$ or $AdS_7 \times S^4$, respectively.

Second, the discretized closed p-brane provides a manifestly weakly coupled partonic description in the tensionless limit. Indeed, taking the *singular tensionless limit*, *viz.*

$$L\mu \to 0 \; , \qquad T_p \mu^{-p-1} \to 0 \; , \tag{1.19}$$

where μ is a lattice constant, and restricting to a sector with a *fixed number N of partons*, we find the $\mathfrak{sp}(2N)$-gauged sigma model [80]

$$S \; = \; \frac{1}{4} \int Y^{IA} D Y_{IA} \; , \tag{1.20}$$

where $I = (i, \xi)$, $\xi = 1, \ldots, N$. In the quantum theory, we impose each $\mathfrak{sp}(2)_{(\xi)}$ strongly on physical states leaving the off-diagonal generators to vanish weakly. Thus, each parton is a scalar singleton on the hypercone (1.2). The global $\mathfrak{sp}(2N)$-invariance enforces the symmetrization leading to (1.18).

However, unless $D = 3 \bmod 4$, the wave-functions exhibit a *global anomaly* under large $Sp(2N)$ gauge transformations in the form of reflections in the apex of the Dirac hypercone. The functions also present subtleties in the form of δ-function distributions, that we conjecture combine with the analytic non-normalizable part into normalizable states – which we think of as squeezed versions of normalizable scalar-field mode-functions in ordinary anti-de Sitter spacetime.

1.4 SINGLETON CLOSED STRING/OPEN MEMBRANE

The natural interpretation is that the multi-singleton system lives in the *phase space* of the Dirac hypercone, where the two-parton sector yields Vasiliev's equations as discussed

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[2]We use the terminology of [40, 70].

above and more generally (1.18) has a natural interpretation as a generalized Chan-Paton factor.

In the continuum limit $N \to \infty$ the discrete tensionless p-brane (1.20) becomes a 2d chiral $\mathfrak{sp}(2)$-gauged phase-space sigma model [24], which we find to be critical in $D = 7$ where it is furthermore dual to a WZW-model based on $\widehat{\mathfrak{so}}(6,2)_{-2}$ in turn containing the coset model

$$\widehat{\mathfrak{so}}(6,2)_{-2}/(\widehat{\mathfrak{so}}(6) \oplus \widehat{\mathfrak{so}}(2))_{-2} \ , \tag{1.21}$$

with vanishing Virasoro charge, giving rise to a chiral ring generated by $\mathfrak{D}$-valued weight-0 spin fields. We identify this topological closed string as the proper framework for computing with the generalized Chan-Paton factor.

We expect the actual space-time dynamics to arise via deformations of the closed string governed by a topological open membrane equipped with a suitable generalization of the bi-local structures of the open string. Its pursuit is a truly challenging problem, whose resolution we believe will contain important new stringy physics.

2 FROM CUSPS TO SINGLETONS

In this Section we analyze anti-de Sitter analogs of states on low Regge trajectories, that is, states with spin S and small excitation energy $E - S$.

We shall first consider the semi-classical representation as long rotating p-branes with energy and spin concentrated to cusps [71, 81, 82, 72], giving rise to wave-functions depending on finite sets of oscillators. Roughly speaking, the physical role of the extended part of the brane is limited to constraining the cusps to angular motion, resulting in a dilute "gas" relatively insensitive to the bare p-brane tension.

To exhibit their "singletonic" nature, we shall then examine an alternative description of the partons as short open p-branes attached to a dual $(D-2)$-brane in AdS_D. This system depends on the asymptotic p-brane tension, rather than the bare tension, and in a decoupling limit [78], where the bare tension vanishes, it becomes a $(D-1)$-dimensional conformal field theory – living on the brane at the end-of-the-universe – where each parton is realized as a proper singleton.

2.1 Partonic Regions of Brane Phase Space

We are interested in the dynamics of closed p-branes described by the ordinary Nambu-Goto action

$$S_{\mathrm{NG}} \ = \ -T_p \int d^{p+1}\sigma \sqrt{-\det g} \ , \tag{2.1}$$

with "bare" tension T_p and induced metric $g_{\alpha\beta} = \partial_\alpha X^m \partial_\beta X^n G_{mn}(X)$, where G_{mn} is the metric of anti-de Sitter spacetime with radius L.

The phase-space has a covariant meaning as the space of all classical solutions, which is suitable for diffeomorphism invariant field theories, since it does not rely on fixing the topology. Moreover, it avoids the strong-coupling problems at small tension that arise

from parameterizing phase space using maps from a smooth p-dimensional volume to target space.

As in ordinary field theory, one then considers solutions with space-time energy-momentum concentrated to portions of the worldvolume that can be made asymptotically relatively small in some parametric limit. Solutions containing N solitons can then be faithfully represented by their positions,

$$\{X^m(\tau;\xi)\}_{\xi=1}^N \ . \tag{2.2}$$

The resulting partonic regions of phase space give rise to sectors of wave-functions of the form

$$\Psi_N(\{X^m(\xi)\}) \ , \tag{2.3}$$

subject to physical-state conditions and Bose symmetry.

We shall consider partons obtained by minimizing the energy E in parameter families of rotating branes while keeping the angular momentum S fixed [71, 72]. The variational parameters represent adiabatic deformations in which the centrifugal force pushes the energy-momentum towards relativistic portions of the worldvolume, where it opens up into a folded shape that we shall refer to as a *cusp*. In general, the cusps attract each other, which is not a problem for stability, however, provided that the system has non-vanishing impact parameters [72].

The motion of the cusps is governed by their relativistic inertia as well as the inward pull from the tension. In global coordinates, with a radius r vanishing at the center-of-mass and $D-2$ angles, this yields angular motion with suppressed radial fluctuations, resulting in that each cusp excites $(D-2)$ degrees of freedom. This behavior is reminiscent of that of anti-de Sitter singletons, and the partonic behavior indeed requires an *enhanced accumulation* of energy-momentum, exhibited in anti-de Sitter spacetime but not in flat spacetime.

The size of the fluctuations around a solution $\overline{X}^m$ is governed by the *effective tension*

$$T_{\text{eff}}L^{p+1} \ = \ T_pL^{p+1}\sqrt{-\det\overline{g}} \ , \tag{2.4}$$

which vanishes on the cusps and is of order T_pL^{p+1} not too far away from them. Thus, in the limit of large T_pL^{p+1} the functional integral collapses to an integral over $X^m(\tau;\xi)$.

This contrasts to ordinary field theories, where non-perturbative degrees of freedom in general decouple in the free limit. A related subtlety presents itself in that the size of the cusps remains finite and of order L, essentially due to the fact that the angular velocity is small and of order $1/L$. However, the important geometric parameter is the ratio between the size and the relative separation of the cusps, which can be made arbitrarily small by considering states with sufficiently large spin. This leads to a partonic region of the p-brane phase space, consisting of well-defined solitons with small and fixed energy $E-S$, which should dominate the classical charges as well as the functional integral.

Having made these preliminary remarks, let us now turn to a more quantitative analysis.

2.2 ROTATING STRINGS

Let us consider the folded and rotating closed string in $(AdS_D)_L$ given by [71]

$$ds^2 = -\cosh^2\frac{r}{L}\,dt^2 + dr^2 + L^2\sinh^2\frac{r}{L}\,d\Omega_{D-2}^2\,, \tag{2.5}$$

$$t = t(\tau)\,, \qquad r = r(\sigma)\,, \qquad \phi = \frac{\omega_0 t}{L}\,,$$

$$\omega_0 = \omega(r_0) \equiv \coth\frac{r_0}{L}\,, \tag{2.6}$$

where ϕ is an azimuthal angle on S^{D-2}; $2r_0$ is the proper length of the string, and the map $r : \sigma \mapsto r(\sigma)$ has winding number 1. The induced worldvolume metric reads

$$ds^2(g) = \mu^2(-dt^2 + dy^2)\,, \qquad dr^2 = \mu^2 dy^2\,, \qquad \mu^2 \equiv \cosh^2\frac{r}{L} - \omega_0^2\sinh^2\frac{r}{L}\,, \tag{2.7}$$

where $y(r) = \int_r^{r_0}\mu^{-1}dr$, that is

$$\frac{dy}{dr} = -\frac{1}{\mu}\,, \qquad e^{\frac{r-r_0}{L}} = \frac{1}{\cosh\frac{y}{L}} + \mathcal{O}(e^{-\frac{2r_0}{L}})\,, \tag{2.8}$$

so that $y(r_0) = 0$ and $y(0) = r_0 + L\log 2 + \mathcal{O}(e^{-\frac{2r_0}{L}})$.

The classical energy and spin are given by[3]

$$E_{\mathrm{cl}} = 4TL\int_0^{r_0+L\log 2} dy\,\cosh^2\frac{r}{L} + \mathcal{O}(e^{-\frac{2r_0}{L}})\,, \tag{2.9}$$

$$S_{\mathrm{cl}} = 4TL\omega_0\int_0^{r_0+L\log 2} dy\,\sinh^2\frac{r}{L} + \mathcal{O}(e^{-\frac{2r_0}{L}})\,. \tag{2.10}$$

For $r_0 \gg L$, a fraction fE_{cl} arises from a small interval of y-values of width $\Delta \sim -\frac{L}{2}\log(1-f) \ll r_0$, i.e. the classical energy and spin are dominated by the contributions from two localized cusps, $\xi = 1, 2$, of fixed width $\Delta \sim L \ll r_0$ for fixed f. The spin and energy are thus given to leading order by a *sum* over contributions from each cusp,

$$E_{\mathrm{cl}} \approx \sum_{\xi=1,2} E(\xi)\,, \qquad S_{\mathrm{cl}} \approx \sum_{\xi=1,2} S(\xi)\,, \tag{2.11}$$

with $E(\xi)$ and $S(\xi)$ determined by the *local* functional form of the energy and spin densities at the cusps. Since these agree to leading order, it follows that

$$E(\xi) = S(\xi)\,, \tag{2.12}$$

and hence that

$$E_{\mathrm{cl}} \approx S_{\mathrm{cl}}\,. \tag{2.13}$$

The proportionality between $E(\xi)$ and $S(\xi)$ implies that the cusps have generalized angular momenta that are "light-like" in the sense that

$$\frac{1}{2}M_A{}^C(\xi)M_{BC}(\xi) = 0\,. \tag{2.14}$$

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[3]Our choices of conventions for $\mathfrak{so}(D-1,2)$ are given in Appendix A.

For later reference, we note that

$$E_{\text{cl}} \approx S_{\text{cl}} \sim TL^2 \, e^{2r_0/L} \ , \tag{2.15}$$

and that the leading correction to the energy is logarithmic in S_{cl}, *viz.*

$$E_{\text{cl}} - S_{\text{cl}} \sim TL^2 \log \frac{S_{\text{cl}}}{TL^2} \ , \tag{2.16}$$

i.e. the cusps interact via a linear potential.

More general spiky string solutions have been given in [72]. These contain N cusps rotating co-planarly, each carrying energy E_{cl}/N and spin S_{cl}/N. These highly symmetric configurations have vanishing impact parameters, and are therefore unstable against perturbations [72]. Solutions with randomly distributed impact parameters will describe stable clusters of cusps, with total angular momentum obeying the addition rule

$$M_{AB} \approx \sum_{\xi=1}^{N} M_{AB}(\xi) \ , \tag{2.17}$$

where $M_{AB}(\xi)$ are light-like angular momenta, and the subleading terms are due to interactions, which we shall comment on below.

In the flat space-time limit $L \to \infty$, one finds scale-invariant energy and spin densities, proportional to $(1 - (r/r_0)^2)^{-1/2}$ and $(r/r_0)^2(1 - (r/r_0)^2)^{-1/2}$, respectively. Hence $\Delta \sim r_0$, so the densities do not form lumps, the energy and spin are related non-linearly as fixed by dimensional analysis, and the partonic picture is lost.

To examine the cusps in more detail, we consider the expansion of $S_{\text{NG}}[X]$ in inverse powers of $TL^2 \gg 1$ around the folded string solution $\overline{X}^m$,

$$S_{\text{NG}}[X] = \bar{S}_{\text{NG}} + \sum_{p=2}^{\infty} S_p[\varphi] \ , \qquad \varphi^a = T^{1/2}(X^m - \overline{X}^m)\overline{E}_m{}^a \ , \tag{2.18}$$

where φ^a is a canonically normalized fluctuation field. The quadratic part can be expressed using normal coordinates as

$$S_2[\varphi] = -\frac{1}{2} \int d^2\sigma \sqrt{-\bar{g}} \left(\nabla^\alpha \varphi^a \nabla_\alpha \varphi_a - E^{\alpha a} E_\alpha{}^b R_{ac,bd} \varphi^c \varphi^d \right. \tag{2.19}$$

$$\left. + \nabla^\alpha \varphi^a \nabla^\beta \varphi^b (2 E_{\alpha[a} E_{\beta b]} - \bar{g}_{\alpha\beta} E^\gamma{}_a E_{\gamma b}) \right) \ ,$$

where ∇_α contains the space-time Lorentz connection and $E_\alpha{}^a = \partial_\alpha \overline{X}^m \overline{E}_m{}^a$. The fluctuations $E_\alpha{}^a \varphi_a$ contain pure-gauge fluctuations and a zero-mode describing changes in the physical length of the string, that we shall not consider here. One finds that $S_2 = S_2[\varphi_{i'}] + S_2[\varphi^{\underline{t}}, \varphi^{\underline{r}}, \varphi^{\underline{\phi}}]$, with $\varphi_{i'}$, $i' = 1, \ldots, D-3$, containing the fluctuations tangent to S^{D-3}, for which $\nabla_\alpha \varphi_{i'} = \partial_\alpha \varphi_{i'}$. The $(\varphi^{\underline{t}}, \varphi^{\underline{r}}, \varphi^{\underline{\phi}})$-sector contains one physical fluctuation field, $\tilde{\varphi}$.

The end-result reads [81, 82]

$$S_2[\varphi] = -\frac{1}{2} \sum_{i=1}^{D-2} \int dt dy \left((\partial \varphi_i)^2 + \frac{q_i \mu^2 + q_i' \mu^{-2}}{L^2}(\varphi_i)^2 \right) + S_2' \ , \tag{2.20}$$

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where $\varphi_i = (\varphi_{i'}, \tilde{\varphi})$ $(i = 1, \ldots, D-2)$; the parameters q_i and q_i' are given by

$$q_{i'} \;=\; 2 \,, \qquad q_{i'}' \;=\; 0 \,, \tag{2.21}$$

$$\tilde{q} \;=\; 4 + \tilde{q}' \,, \tag{2.22}$$

with $\tilde{q} = q_{D-2}$ and $\tilde{q}' = q_{D-2}'$; and S_2' contains gauge artifacts and the radial zero-mode. The parameter $\tilde{q}'$ is the coefficient of the $\mathcal{R}_{(2)}$-term, which receives corrections from the conformal anomaly of the Nambu-Goto action, as explained in [81], where it was also proposed that $\tilde{q}' = 0$ in the case of critical strings.

Expanding the physical fields as

$$\varphi_i \;=\; \sum_k e^{\frac{i\nu_{i,k}t}{L}} \varphi_{i,k}(y) \frac{a_{i,k}^\dagger}{\sqrt{\nu_{i,k}}} + \text{h.c.} \,, \tag{2.23}$$

where we have assumed that all frequencies are non-vanishing, one finds that the mode functions obey the Schrödinger problem $(-r_0 \leq r \leq r_0)$

$$\left(-L^2 \frac{d^2}{dy^2} + q_i\mu^2 + q_i'\mu^{-2}\right)\varphi_{i,k} \;=\; \nu_{i,k}^2 \varphi_{i,k} \,, \qquad \left.\frac{d\varphi_{i,k}}{dy}\right| = 0 \,. \tag{2.24}$$

In the case $q_i' = 0$, there are two small potential wells of width L centered around $r = \pm r_0$, that become well-separated for $r_0/L \gg 1$, and reduce to exactly solvable *Pöschl-Teller potentials* [83][4] in the limit $r_0/L \to \infty$,

$$q_i\mu^2 \;=\; q_i\left(1 - \frac{1}{\cosh^2 \frac{y}{L}}\right) + \mathcal{O}(e^{-2r_0/L}) \,. \tag{2.25}$$

For $q \leq 6$ this potential admits precisely one even bound state, given by

$$\varphi \;=\; \mathcal{N}\left(\frac{1}{\cosh \frac{y}{L}}\right)^{\nu^2} \,, \qquad \nu^2 \;=\; \frac{1}{2}(\sqrt{1+4q} - 1) \,, \tag{2.26}$$

where the normalization $\mathcal{N} \sim L^{-1/2}$. Thus, the frequency of the *overall-transverse bound-state modes* is given by

$$\nu_{i'} \;=\; 1 \,, \tag{2.27}$$

The longitudinal bound-state problem for $i = D-2$ is highly sensitive to the precise value of the parameter $\tilde{q}'$: a negative value deepens the potential well so that there may arise additional bound states, while a positive value makes the well more shallow so that bound states may be lost. Assuming $\tilde{q} = 4$ we find the *longitudinal bound-state mode*

$$\tilde{\nu}^2 \;=\; \frac{1}{2}(\sqrt{17} - 1) \,. \tag{2.28}$$

---

[4]The Hamiltonian has a superpotential, $H = A_{\alpha_0}^+ A_{\alpha_0}^- + C_0$ with $A_\alpha^\pm = \mp\frac{d}{dy} + W_\alpha$, $W_\alpha = \alpha \tanh y$, $\alpha_0 = C_0 = (\sqrt{1+4q} - 1)/2$, where the oscillators obey the deformed Heisenberg relation $A_\alpha^- A_\alpha^+ = A_{f(\alpha)}^+ A_{f(\alpha)}^- + R_\alpha$, with $f(\alpha) = \alpha - 1$ and $R_\alpha = 2\alpha - 1$. The normalizable solutions to $(H - \lambda)\varphi = 0$ are given by $\varphi_{(n)} = \mathcal{N}_n A_{\alpha_0}^+ A_{f(\alpha_0)}^+ \cdots A_{f^{n-1}(\alpha_0)}^+ \exp(-\int^y W_{f^n(\alpha_0)})$ and $\lambda_n = C_0 + R_{\alpha_0} + \cdots + R_{f^{n-1}(\alpha_0)}$ for $n = 0, 1, \ldots, [\alpha_0 - 1]$.
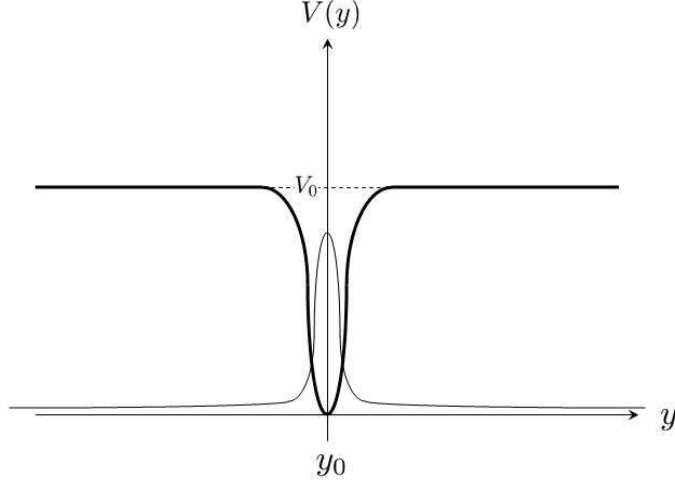
15

Figure 1: *The Pöschl-Teller potential*: The σ-dependent mass-term on the rotating string has height $\sim 1/L$ a small "well" of width $\sim L$ which fits precisely one even bound state.

For finite r_0/L, the bound-state wave functions $\varphi(\xi)$ ($\xi = 1, 2$) have finite small overlaps. As a result, the Hamiltonian is diagonalized by wave functions of the approximate form $\varphi_\pm = \frac{1}{\sqrt{2}}(\varphi(1) \pm \varphi(2))$, while there are only small corrections to the frequencies.

The *wave modes* are approximately given by

$$\varphi_{i,k} \approx \mathcal{N}_{i,k} \, \cos \frac{ky}{r_0} \, , \qquad \nu_{i,k}^2 \; = \; q_i + \frac{k^2 L^2}{r_0^2} \, , \qquad k = 0, 1, \dots \, , \qquad (2.29)$$

with $\mathcal{N} \sim r_0^{-1/2}$, and can be divided into *short waves* with $k \gg r_0/L$, $\nu_k \approx kL/r_0$, that decouple in the limit $r_0/L \to \infty$; *long waves* with $k \ll r_0/L$, $\nu_k \approx \sqrt{2}(1 + k^2 L^2/4r_0^2)$; and *intermediate waves* with $k \sim r_0/L$, $\nu_k \sim 1$. We note that both long and intermediate waves have frequencies approximately equal to the excitation energy of the Pöschl-Teller bound states.

Analogously, the normal-coordinate fluctuations around the symmetric N-cusp solutions [72] give rise to $N(D-2)$ bound-state oscillators $a_i^\dagger(\xi)$, $\xi = 1, \dots, N$, splitting into $N \times (D-3)$ overall-transverse states with frequency 1, and N longitudinal states with frequency $\tilde{\nu}$. Thus the N-cusp ground-state energy, $E = \sum_{n=0}^{\infty} E_n$, with $E_0 = E_{\mathrm{cl}}$ and $E_n = (TL^2)^{1-n}\mathcal{E}_n$, receives a 1-loop cusp correction, $E_1 = E_1(\text{waves}) + E_1(\text{cusps})$, given by the zero-point energy contribution

$$E_1(\text{cusps}) \; = \; N \times (D - 3 + \tilde{\nu}) \times \frac{1}{2} \, . \qquad (2.30)$$

Remarkably, even though we are examining the string in the limit $TL^2 \gg 1$, the contribution from each cusp is given by the singleton ground-state energy ϵ_0 given in (1.3), plus a finite anomalous part[5].

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[5]From the holographic point-of-view, this indicates that insertions of gauge-covariant $D_\perp$ derivatives into light-like bilinear Wilson lines W receive only very mild anomalous corrections at strong coupling: if W is built from two scalar fields, S light-like D_+, K $D_\perp$, and $\widetilde{K}$ D_-, then the string result indicates that $\Delta(W) - (D - 3 + S + K + \widetilde{K}) = \tilde{\nu} + (\tilde{\nu} - 1)\widetilde{K} + (f_1(\lambda) + f_2(\lambda)\frac{1}{S} + \cdots)\log S + g_1(\lambda; K)\frac{1}{S} + g_2(\lambda; K)\frac{1}{S^2} + \cdots$.

2.3 DILUTE-GAS APPROXIMATION

In this section we provide a heuristic interpretation of the results obtained in the previous section. To begin with, in (2.18) the interactions contained in S_p for $p > 2$ are built from multiple gradients $\partial_\alpha \overline{X}^m$ and $\overline{\nabla}_\alpha \varphi_i$ contracted with $\bar{g}^{\alpha\beta}$, and blow up inside the Pöschl-Teller potential wells. Combined with the variational principle, this results in modified boundary conditions, so that actual physical quantities remain finite in the classical theory.

Alternatively, the interactions[6] can be obtained using the variational parameters, *e.g.* by considering folded strings of length l rotating with frequency $\omega < \omega(l)$ such that $\mu(r,\omega)| = \mu(l,\omega) > 0$. We note that fixing S_{cl} yields a line $\omega = \omega(l, S_{\mathrm{cl}})$ in the region $\omega > \omega_0$ and $l < r_0$, terminating at the point $\omega = \omega_0$ where E_{cl} becomes minimal. We shall assume that S_p collapses in the limit $TL^2 \to \infty$ and $\mu(l,\omega) \to 0$, to an integral *localized to the cusps* resulting in cusp-wave, wave-wave and cusp-cusp interactions with couplings whose (r_0/L)-dependence is governed by localized integrals of products of mode functions.

The resulting cusp-wave and wave-wave couplings scale like negative powers of $r_0/L \sim \log S_{\mathrm{cl}}$, such that at fixed order in number of oscillators, the wave-wave vertices are suppressed by larger powers than wave-cusp vertices. The cusp-cusp couplings scale like $\exp(-d/L) \sim (S_{\mathrm{cl}})^{-d/r_0}$, where d is the distance d between the cusps. The normal-coordinate theory therefore contains a multi-cusp system with Hamiltonian

$$H_N[\{a_i(\xi), a_i^\dagger(\xi)\}_{\xi=1}^N] \,, \tag{2.31}$$

consisting of kinetic terms and an effective long-range potential representing the exchange of long and intermediate wave quanta, and giving rise to a dispersion relation of the form (2.17).

Let us compare the corresponding multi-cusp states,

$$|\Psi\rangle_N \;=\; \Psi[\{a_i^\dagger(\xi)\}_{\xi=1}^N]|0\rangle \,, \tag{2.32}$$

with the tensor products of scalar-singleton states belonging to a sector carrying one large spin component, say $|\Psi\rangle \in \mathfrak{D}$ obeying

$$\langle\Psi|M_{D-2,D-1}|\Psi\rangle \sim S_{\mathrm{cl}} \gg 1 \,. \tag{2.33}$$

Defining $A_{i'} \equiv L_{i'}^-/\sqrt{2S_{\mathrm{cl}}}$, $i' = 1, \ldots, D-3$, and $A_\pm \equiv (L_{D-2}^- \pm iL_{D-1}^-)/\sqrt{8S_{\mathrm{cl}}}$, where $L_r^\pm$ $(r = 1, \ldots, D-1)$ are the spin-boost generators defined in (A.8) and (A.9), then the scalar-singleton mass-shell condition $L_r^+ L_r^+ |\Psi\rangle = 0$ corresponding to the singular vector (A.16), yields

$$\left(4A_-^\dagger A_+^\dagger + \sum_{i'=1}^{D-3}(A_{i'}^\dagger)^2\right)|\Psi\rangle \;=\; 0 \,. \tag{2.34}$$

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[6]The normal-coordinate Hamiltonian does not incorporate splitting and joining of strings, which are processes suppressed by powers of the closed-string coupling g_s. In the case of spiky strings, the underlying "mechanical" forces are largest in the interior, which is where they tend to break most easily [84, 72]. Thus, the first subleading order in g_s contains processes in which N-cusp solitons $\leftrightarrow (N_1, N_2)$-cusp two-body solitons with $N_1 + N_2 = N + 2$ (see Fig. 2).
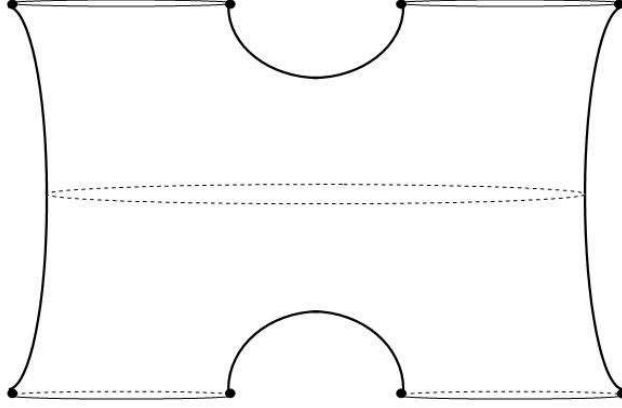
Figure 2: *Long-string interactions*: Two long strings with cusps at their ends interact. Two cusps annihilate while emitting waves that later recombine into a new pair of cusps.

Thus, acting in the large-spin sector, $A_i \equiv (A_{i'}, A_+)$ behave as oscillators,

$$[A_i, A_j^\dagger] \approx \delta_{ij} \ , \tag{2.35}$$

while A_- behaves classically

$$[A_-, A_-^\dagger] \ \sim \ 1/S_{\mathrm{cl}} \ , \tag{2.36}$$

and decouples in the semi-classical limit via the mass-shell condition (2.34). In the normal-coordinate expansion, $(E - M_{D-2,D-1})/L$ is identified as the world-sheet Hamiltonian, although E and $M_{D-2,D-1}$ are not quantized separately [81, 82]. The energy and spin eigenvalues carried by $A_{i'}^\dagger$ and $A_+^\dagger$ correspond to frequencies $\nu_{i'} = 1$ and $\nu_+ = 0$, respectively. Thus, in going from strong to weak tension, we may identify $A_i^\dagger$ as the limit of $a_i^\dagger$, where we note that the frequency $\nu_{i'}$ of the overall transverse set of oscillators remain constant while the frequency of the longitudinal oscillator changes from $\tilde{\nu}$ to ν_+.

Hence, the state space of the multi-cusp system can be identified as a deformed corner of a multi-singleton system[7], schematically

$$\mathcal{S}_{\mathrm{cusps}} \ \subset \ \bigoplus_{N=2}^{\infty} m_Y \left[\mathfrak{D}^{\otimes N} \right]_Y \ , \tag{2.37}$$

where Y denote Young tableaux and m_Y multiplicities. At the linearized level, the Bose symmetry of the normal-coordinate quantum field theory implies that products of $a_i^\dagger(\xi)$ are fully symmetrized, *i.e.* at this level $m_{\mathrm{symm}} = 1$ while all other Young projections are absent. At finite tension and including interactions, or in the corresponding discretized

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[7]Pulsating strings, with small S_{cl} an σ-independent effective tension T_{eff}, should correspond to deformations of multi-singleton states, though here the anomalous corrections to the energy will be much larger.

quantum mechanical formulation, which we shall study below, it is natural to expect further states that break S_N to $\mathbb{Z}_N$. Indeed, the spectrum of the $10D$ tensile IIB closed string in flat spacetime appears to cross over into a tensionless spectrum on $\text{AdS}_5 \times S^5$ given by $\mathbb{Z}_N$-invariant tensor products of $\mathfrak{psu}(2,2|4)$ supersingletons (corresponding holographically to single-trace operators) [68, 33, 37, 40]. However, as we shall see, the cyclic invariance seems to be associated to the stringy nature of the tensile deformation, rather than to any intrinsic property of the undeformed tensionless theory, which we shall find consists of fully symmetrized states – at least in the scheme for discretization that we set up.

The fact that each string-cusp yields $D - 2$ oscillators is a direct result of the singleton-like rotational motion with heavily suppressed radial fluctuations, rather than of the fact that the string has $D - 2$ transverse directions. This suggests that multi-singleton states arise also for higher p-branes.

2.4 ROTATING MEMBRANES AND p-BRANES

A closed membrane with topology $T^2 \times \mathbb{R}$, parameterized by $(\tau; \sigma, \rho) \in \mathbb{R} \times [0, 2\pi[\times [0, 2\pi[$, can be folded and embedded into the following rotating configuration in $(AdS_D)_L$ [37]

$$ t \;=\; t(\tau) \,, \qquad r \;=\; r(\sigma, \rho) \,, \qquad \theta \;=\; \theta(\rho) \,, \qquad \phi \;=\; \omega_0 t \,, \qquad (2.38) $$

where θ is a polar angle in S^{D-2}, defined by $d\Omega^2_{D-2} = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\Omega^2_{D-4}$, and the maps $r(\cdot, \rho) : \sigma \mapsto r(\sigma, \rho)$ and $\theta : \rho \mapsto \theta(\rho)$ have winding number 1. The embedding depends on two variational parameters, parameterizing the turning points of θ and r, so that $\theta \in [\frac{\pi}{2} - \frac{\ell}{2L}, \frac{\pi}{2} + \frac{\ell}{2L}]$ and $r \in [-r_0, r_0]$ when $\theta = \pi/2$. These parameters are fixed by minimizing the energy at fixed spin under the assumption that the membrane has two long edges at $\theta = \frac{\pi}{2} \pm \frac{\ell}{2L}$ and two short edges at $r = \pm r_0$, viz.

$$ r_0 \gg L \,, \qquad \ell \sinh \frac{r_0}{L} \ll L \,, \qquad (2.39) $$

and that the induced volume element, given by

$$ -\det g = \left(\frac{\partial r}{\partial \sigma}\right)^2 \left(\cosh^2 \frac{r}{L} - \omega^2 \sinh^2 \frac{r}{L} \sin^2 \theta\right) \left(L^2 \sinh^2 \frac{r}{L} \left(\frac{\partial \theta}{\partial \rho}\right)^2 - \left(\frac{\partial r}{\partial \rho}\right)^2\right) \quad (2.40) $$

vanishes on the short edges. The latter condition requires $\coth \frac{r|}{L} = \omega \sin \theta$, where $r|$ is the restriction of r to the short edges, that in turn implies $\frac{\partial r|}{\partial \rho} = -\omega L \sinh^2 \frac{r|}{L} \cos \theta \frac{\partial \theta}{\partial \rho}$, which together with the shortness condition yields

$$ \det g \approx -\mu^2 \left(\frac{\partial r}{\partial \sigma}\right)^2 L^2 \sinh^2 \frac{r}{L} \left(\frac{\partial \theta}{\partial \rho}\right)^2 \,. \qquad (2.41) $$

Thus, to the leading order one can set

$$ r \;\approx\; r(\sigma) \,, \qquad (2.42) $$

and expand around $\theta = \frac{\pi}{2}$, resulting in

$$E_{\text{cl}}(\ell, r_0) \;=\; 8T\ell L^2 \int_0^{r_0 + L \log 2} dy \; \sinh \frac{r}{L} \cosh^2 \frac{r}{L} \; , \tag{2.43}$$

$$S_{\text{cl}}(\ell, r_0) \;=\; 8T\ell L^2 \omega \int_0^{r_0 + L \log 2} dy \; \sinh^3 \frac{r}{L} \; . \tag{2.44}$$

The leading contributions scale like

$$E_{\text{cl}}(\ell, r_0) \;\approx\; S_{\text{cl}}(\ell, r_0) \sim TL^2 \, \ell \, e^{\frac{3r_0}{L}} \; , \tag{2.45}$$

and the first "anomalous" correction to the energy at fixed spin and variational parameter ℓ scales like

$$E_{\text{cl}}(\ell) - S_{\text{cl}} \;\sim\; (T\ell L^2)^{\frac{2}{3}} S_{\text{cl}}^{\frac{1}{3}} \; . \tag{2.46}$$

It follows that $E_{\text{cl}}(\ell)$ is minimized at fixed S_{cl} in the limit[8]

$$\frac{\ell}{L} \to 0 \; , \qquad \frac{r_0}{L} \to \infty \; , \tag{2.47}$$

where we note that the shortness condition is indeed obeyed, that is

$$\ell \sinh \frac{r_0}{L} \sim S_{\text{cl}} \exp(-2r_0/L)/TL^3 \to 0 \; . \tag{2.48}$$

Hence, the membrane collapses to an infinitely long string-like configuration with

$$E_{\text{cl}} \;=\; S_{\text{cl}} \; . \tag{2.49}$$

Thus, the two-dimensional gas of cusps on the membrane has a vanishing potential to the leading order. This is to be contrasted with the linear potential that arise to the same order in the case of the one-dimensional gas of cusps on the string.

Turning to the fluctuation analysis, the sector of ρ-*independent* fluctuations $\phi_a^{(0)}$, which are normalized with string tension $T\ell$, contains $D-2$ physical fields: $D-4$ overall transverse fields $\varphi_\perp^{(0)}$; an admixture $\tilde{\varphi}^{(0)}$ arising in the $(\varphi^{\underline{t}}, \varphi^{\underline{r}}, \varphi^{\underline{\phi}})$-sector; and $\varphi_{\underline{\theta}}^{(0)}$, which describes an actual deformation of the embedding in target space that cannot be gauged away. To show this one observes that the fluctuation field $\varphi^{\underline{\theta}}$ transforms under world-volume diffeomorphisms with parameters $V = V^\alpha \partial_\alpha$ as $\delta\varphi^{\underline{\theta}} = V^\rho \partial_\rho \theta L \sinh \frac{r}{L}$. There is no ρ-independent mode in $\partial_\rho \theta$, since natural boundary conditions imply that this field vanishes on the long edges. Thus, the variation $\delta\varphi^{\underline{\theta}}$ does not contain any ρ-independent mode, so that $\varphi_{\underline{\theta}}^{(0)}$ cannot be gauged away.

The quadratic action for $\varphi_\perp^{(0)}$ reads

$$S_2[\varphi_\perp] \;=\; -\frac{1}{2} \int dt dy \sinh \frac{r}{L} \left((\partial\varphi_\perp^{(0)})^2 + \frac{3\mu^2}{L^2} (\varphi_\perp^{(0)})^2 \right) \; . \tag{2.50}$$

[8]From the holographic point of view, $r_0 \to \infty$ a UV limit, and $\ell \sinh \frac{r_0}{L} \to 0$ is limit that localizes the Wilson surface coupling to the boundary of the open membrane.

Rescaling, $\varphi_\perp^{(0)} = \hat{\varphi}/\sqrt{\sinh(r/L)}$, expanding $\hat{\varphi} = \sum_k e^{\frac{i\nu_k t}{L}} \hat{\varphi}_k(y) a_k^\dagger/\sqrt{\nu_k}$, and taking the limit $r_0/L \to \infty$ using that $\sinh(r/L) \approx \exp(r_0/L) \cosh^{-1}(y/L)$ and $\coth(r/L) \approx 1 + 2\exp(-2r_0/L)\cosh^{-2}(y/L)$, one finds the Pöschl-Teller potential problem

$$\left(-L^2\frac{d^2}{dy^2} + q\mu^2\right)\hat{\varphi}_k = (\nu_k^2 + \tfrac{1}{2})\hat{\varphi}_k , \qquad \frac{d\hat{\varphi}}{dy}\Big|_{y=0} = 0 , \qquad q = \frac{15}{4} , \qquad (2.51)$$

that admits precisely one bound state with protected frequency

$$\varphi_\perp^{(0)} = \mathcal{N}\frac{1}{\cosh\frac{y}{L}} , \qquad \nu_\perp^2 = 1 , \qquad (2.52)$$

where the power of $e^{-\frac{r_0}{L}}$ in $(\sinh\frac{r}{L})^{-\frac{1}{2}}$ has been absorbed into the normalization. Not so surprisingly, this solution is identical to that found in the string case. We expect that also $\varphi_{\underline{\theta}}^{(0)}$ contains the same bound state, so that there is a total of $D - 3$ bound states at each cusp with frequency $\nu = 1$, while $\tilde{\varphi}^{(0)}$ should contain bound states with anomalous frequency, c.f. eqs. (2.22) and (2.28).

In the case of a folded and rotating p-brane with $p \leq D - 2$, the topology is $T^p \times \mathbb{R}$, and the folded world-volume extends in one radial direction and $p - 1$ directions transverse to the plane of rotation. As for $p = 2$, the classical energy is minimized on a stringy configuration, with

$$E_{\mathrm{cl}} = S_{\mathrm{cl}} \sim T_p L^2 \ell^{p-1} e^{\frac{(p+1)r_0}{L}} , \qquad E_{\mathrm{cl}}(\ell) - S_{\mathrm{cl}} \sim (T_p \ell^{p-1} L^2)^{\frac{2}{p+1}} S_{\mathrm{cl}}^{\frac{p-1}{p+1}} . \qquad (2.53)$$

The quadratic action for the $D - p - 2$ overall transverse "stringy" fluctuations read

$$S_2 = -\frac{1}{2}\int dt\, dy \sinh^{p-1}\frac{r}{L}\left((\partial\varphi_\perp^{(0)})^2 + \frac{(p+1)\mu^2}{L^2}(\varphi_\perp^{(0)})^2\right) . \qquad (2.54)$$

This is a Pöschl-Teller problem for $(\sinh\frac{r}{L})^{\frac{p-1}{2}}\varphi_\perp^{(0)}$ with $q = \frac{1}{4}(p^2 + 4p + 3)$ and eigenvalue $\nu^2 + \frac{p-1}{2}$. There are $[(p-1)/2]$ bound states in the potential well. Thus, for $p \geq 3$, there is at least one extra bound-state oscillator, over and above the ground state, making the interpretation of the cusps more problematic. We shall address this issue below in Sections 2.6, in relation to branes at the end-of-the-universe, and in Section 3.3, by examining tensile deformations of the tensionless discretized brane.

Thus the covariant phase-space of p-branes in anti-de Sitter spacetime contains partonic regions giving rise to wave functions that fit into large-spin corners of multi-singleton spaces. Remarkably, the quantum numbers remain protected in the bosonic case, although supersymmetry should ultimately be of importance.

2.5 Remarks on The Superstring and Supermembrane

In the case of the superstring on $AdS_5 \times S^5$ [71, 81, 82], the scalar fluctuations in S^5 have vanishing mass and thus do not give rise to any bound states. The fermionic fluctuations are $4 + 4$ Majorana spinors Θ with mass $\epsilon\frac{\mu}{L}$, $\epsilon = \pm 1$, [81],

$$S_2[\Theta] = i\int dt\, dy\left(\bar{\Theta}\gamma^\alpha\partial_\alpha\Theta + \frac{\mu}{L}\bar{\Theta}\Theta\right) . \qquad (2.55)$$

Taking $\gamma^\alpha = (i\sigma^2, \sigma^1)$ and $C = -i\sigma^2$ and real Θ, the natural boundary conditions $\bar{\Theta}\gamma^\alpha n_\alpha \delta\Theta| = 0$ at $r = \pm r_0$ imply $(U\delta U - V\delta V)| = 0$ where $U = \Theta_1$ and $V = \Theta_2$. This leaves two possibilities that do not violate the discrete $\mathbb{Z}_2$ symmetry under exchange of cusps, namely $U| = \sigma V|$ for $\sigma = \pm 1$. Expanding into mode-functions $(e^{\frac{i\nu_k t}{L}} U_k + \text{h.c.})$ and $(e^{\frac{i\nu_k t}{L}} V_k + \text{h.c.})$, the Dirac equation reads

$$\left(L\frac{d}{dy} + i\nu\right)U_k + \epsilon\mu V_k = 0 , \qquad \left(L\frac{d}{dy} - i\nu\right)V_k + \epsilon\mu U_k = 0 . \qquad (2.56)$$

Using $dr^2 = \mu^2 dy^2$, one finds bound states peaked at $r = \sigma r_0$ given by

$$U = \mathcal{N} e^{\frac{(\sigma r - r_0)}{L}} , \qquad \nu = 0 , \qquad (2.57)$$

where $\mathcal{N} \sim L^{-1/2}$. Thus, there are $4 + 4$ *real* Clifford-algebra elements localized at each cusp, that can be combined into $2 + 2$ fermionic oscillators[9] $\alpha_I^\dagger$ and $\beta_{I'}^\dagger$, $I, I' = 1, 2$. Since $\nu = 0$, they do not contribute to the zero-point energy.

The wave functions at each cusp are thus in rough agreement with a large-spin sector of the $\mathfrak{psu}(2, 2|4)$ singleton, in which the spin-boosts and the supercharges behave asymptotically as the bound-state oscillators.

In the case of the supermembrane on $(AdS_7)_L \times (S^4)_{L/2}$ the ρ-independent fluctuations in S^4 are described by the action $-\frac{1}{2}\int dtdy \sinh\frac{r}{L} (\partial\varphi^{(0)})^2$, which corresponds to a Pöschl-Teller problem for $(\sinh\frac{r}{L})^{\frac{1}{2}}\varphi^{(0)}$ with $q = \frac{3}{4}$ and eigenvalue $\frac{1}{L^2}(\nu^2 + \frac{1}{2})$. This problem admits the solution $\nu = 0$ and $\varphi = 1$, which does not localize on the cusps, as expected. We therefore expect that the fermionic fluctuations contain bound states with $\nu = 0$ corresponding to a total of $2 + 2$ fermionic creation operators, so that the bound-state wave function can be interpreted as the large-spin limit of the $\mathfrak{osp}(8^*|4)$ supersingleton.

2.6 BRANES AT THE END-OF-THE-UNIVERSE

While it has been assumed throughout the above analysis that $TL^2 \gg 1$, it seems natural – drawing on analogies with ordinary soliton quantum mechanics – that the complete string spectrum should be identified with a singleton "gas" in the tensionless regime $TL^2 \ll 1$. This physical picture seems clearer in the case of the membrane, where there is no linear potential and Bose symmetry is more appropriate. An alternative way of thinking of the tensionless limit, is to view the cusps as "mechanical" lumps, and seek an interpretation in terms of short open p-branes ending on a dual $(D - 2)$-brane. This system can then be followed to the boundary region of AdS spacetime, where the effective open-brane tension diverges while the "bare" p-brane tension vanishes, as we shall demonstrate next using some sample calculations.

2.6.1 GIANT VACUUM AND SINGULAR CFT

The spiky strings may be thought of as bound states formed by *long* open strings carrying $U(1)$ Chan-Paton factors. When the physical sizes of the bound states increase – as a

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[9]The wave-modes with $\nu > 0$ gives rise to $4 + 4$ fermionic creation operators for each value of σ.

result of increasing the ratio between spin and tension – it is likely for them to disintegrate, either by formation of cusp-anti-cusp pairs leading to multi-body interactions of the type depicted in Fig. 2, or via processes in which a single cusp is shredded off while the remaining stretched junction to the interior of the string retracts smoothly. Drawing on the classical mechanical properties of branes, one may argue that the latter processes should start to dominate as the bare tension decreases, since then the "snapping" associated with breaking of strings, which is the mechanism responsible for forming cusp-anti-cusp pairs, becomes less distinct.

The released cusp is a lump of relativistic energy moving along a light-like geodesic line. In Poincaré coordinates, *viz.*

$$ds^2 = L^2 \left(u^2 dx^\mu dx_\mu + \frac{du^2}{u^2} \right) , \tag{2.58}$$

the geodesic can be chosen to be a curve with constant $u = u_0$ along the null-direction in x-space. Furthermore, it is natural to assume that the number of degrees of freedom that go into the cusp wave-function should not change in the process. Assuming the wave-function to be built from $(D-2)$ oscillators $a_i^\dagger(\xi)$, this suggests that the released cusp be identified as a *short* open string on a stable $(D-2)$-brane at $u = u_0$. This requires an asymptotically large effective open-string tension

$$T_{\text{s,eff}} = TL^2 u_0^2 \gg \frac{1}{L^2} , \tag{2.59}$$

and that the $(D-2)$-brane couples magnetically to the cosmological constant Λ, that is $\Lambda = \frac{1}{2D!}|dC_{D-1}|^2$ and

$$S_{D-2} = -T_{D-2} \int d^{D-1}\sigma \sqrt{-\det g} + Q_{D-2} \int C_{D-1} , \tag{2.60}$$

with charge and tension given by[10]

$$Q_{D-2} = \left(\frac{D-1}{2(D-2)} \right)^{\frac{1}{2}} T_{D-2} , \qquad K \equiv T_{D-2}L^{D-1} \gg (TL^2)^{\frac{D-1}{2}} . \tag{2.61}$$

The open-string corrections are dressed with positive powers of $1/T_{\text{s,eff}}$ and a suitable open-string coupling $G_{\text{OS,eff}}^2$, assumed to be proportional to the closed-string coupling g_s.

In static, or "Monge"[11], gauge

$$x^\mu = \sigma^\mu , \qquad \phi \equiv w - w_0 = \phi(x) , \qquad w \equiv u^{\frac{D-3}{2}} , \tag{2.62}$$

the limit

$$TL^2 \to 0 , \qquad \text{at fixed } \phi, \, \partial_\mu \phi, \, T_{\text{s,eff}}L^2 , \tag{2.63}$$

---

[10]In the purely bosonic case, we dualize the cosmological constant using the convention $S_{\text{grav}} = -\int d^D X \sqrt{-G}(R + \Lambda)$ with $\Lambda = \frac{1}{2D!}|dC_{D-1}|^2 = -\frac{(D-1)(D-2)}{L^2}$. In the supersymmetric extensions this relation is modified.

[11]Coined by P. Howe.

yields

$$S_{D-2} = -\frac{\widetilde{K}}{2} \int d^{D-1}x \, (\partial\phi)^2 \, , \qquad \widetilde{K} = \frac{4K}{(D-3)^2} \, , \tag{2.64}$$

with evanescent terms given by positive powers of $\partial_\mu\phi/(u_0 w_0)$. Sending also

$$P_\mu/T_{\text{s,eff}} \to 0 \, , \quad G_{\text{OS}} \to 0 \, ; \qquad \text{at fixed } P_\mu \text{ and } K \gg 1 \, , \tag{2.65}$$

where P_μ is the scalar field UV cut-off, yields a decoupled *singular conformal field theory* [78]. This theory is based on the scalar singleton $\mathfrak{D}$ arising from the quadratic action, and can be used to build composite operators (vertices) filling the spectrum

$$\mathcal{S}_{\text{giant}} = \sum_{N=1}^{\infty} (\mathfrak{D}^{\otimes N})_{\text{symm}} \, , \tag{2.66}$$

which we identify as (2.37) with an additional single singleton sector.

On the closed-string side, the limit corresponding to (2.65) is

$$TL^4 u^2 \to \infty \, , \quad g_{\text{s}} \to 0 \, ; \qquad \text{with fixed } E, \, T_{D-2}L^{D-1} \gg 1 \, . \tag{2.67}$$

Drawing on the relation between tension and dilaton in $AdS_5 \times S^5$, this indeed requires $TL^2 \to 0$, which together with the insensitivity of the singular theory to the bare tension TL^2 provides further evidence for that there is a smooth transition from the tensile string to a multi-singleton theory.

As already indicated, supersymmetry will be crucial in order to make the above parametric relations precise. Moreover, the identification of the singleton and the massless sector of the open string on the $(D-2)$-brane in AdS_D is quite cumbersome. The $U(1)$ Chan-Paton factors yield even and odd spins in ordinary flat space, and one might speculate that the tachyon becomes massless as an effect of finite or even vanishing L. In this respect, the identification of the $\mathfrak{psu}(2,2|4)$ supersingleton starting from a $D3$-brane in $AdS_5 \times S^5$ should be is simpler. Finally, from the analysis of Section 2.4, it is clear that the assumption that the wave-functions of the cusps are built from $D-2$ oscillators becomes problematic for $p > 2$.

The above considerations motivate a closer look at the maximally supersymmetric cases where open Type IIB strings end on D3-branes and open M2-branes end on M2 or M5-branes.

2.6.2 Maximally Supersymmetric Cases

Let us discuss the singular limit of maximally supersymmetric M2, D3 and M5 branes in $(AdS_D)_L \times (S^{\widetilde{D}})_{\widetilde{L}}$ with

$$\frac{\widetilde{L}}{L} = \frac{D-1}{\widetilde{D}-1} = \frac{D-3}{2} \, . \tag{2.68}$$

We start from the potentials

$$C_{D-1} = \widetilde{L}^{D-1} u^{D-1} d^{D-1}x \, , \qquad d^D x \equiv dx^0 \wedge \cdots \wedge dx^{D-1} \, , \tag{2.69}$$

$$\widetilde{C}_{\widetilde{D}-1} = L^{\widetilde{D}-1} \omega_{\widetilde{D}-1} \, , \tag{2.70}$$

where $d\omega_{\widetilde{D}-1}$ is the volume form on the unit $S^{\widetilde{D}}$. In Monge gauge, the bosonic fields are $\widetilde{D}+1$ scalar fields w^I defined by

$$\sum_{I=1}^{\widetilde{D}+1}(dw^I)^2 = dw^2 + w^2 d\Omega_{\widetilde{D}}^2 , \qquad w = u^{\frac{\widetilde{L}}{L}} , \tag{2.71}$$

and the fields strengths

$$D3 \quad : \qquad f = da , \qquad\qquad a = dx^\mu a_\mu , \tag{2.72}$$

$$M5 \quad : \qquad h = db + \sqrt{T_5}\widetilde{C}_3 , \qquad b = \frac{1}{2}dx^\mu \wedge dx^\nu b_{\mu\nu} . \tag{2.73}$$

The Lagrangians are given by

$$\mathcal{L}_{D-2} = \mathcal{L}_{\mathrm{NG}} + \mathcal{L}_{\mathrm{tensor}} + \mathcal{L}_{\mathrm{el}} + \mathcal{L}_{\mathrm{magn}} , \tag{2.74}$$

where $g_{\mu\nu} = \widetilde{L}^2 w^{\frac{2L}{L}}\eta_{\mu\nu} + \frac{L^2}{w^2}\partial_\mu w^I \partial_\nu w^I$, so that

$$\mathcal{L}_{\mathrm{NG}} = -T_{D-2}\,\widetilde{L}^{p+1}\left(w^{\frac{DL}{L}} + \frac{L^2}{2\widetilde{L}^2}\eta^{\mu\nu}\partial_\mu w^I \partial_\nu w^I + \cdots\right) , \tag{2.75}$$

and the tensor-field Lagrangians and Wess-Zumino terms are given by [12]

$$D3 \quad : \quad \mathcal{L}_{\mathrm{tensor}} = \sqrt{\det\delta_\nu^\mu + 2\pi\alpha' g^{\mu\rho}f_{\rho\nu}} = 1 + \frac{(2\pi\alpha')2}{4\widetilde{L}^4 w^4}f^{\mu\nu}f_{\mu\nu} + \cdots , \tag{2.77}$$

$$M5 \quad : \quad \mathcal{L}_{\mathrm{tensor}} = \sqrt{1 + \Phi(g,h)} = 1 + \frac{T_5^{-1}}{24\widetilde{L}^6 w^3}h^{\mu\nu\rho}h_{\mu\nu\rho} + \cdots , \tag{2.78}$$

$$\mathcal{L}_{\mathrm{el}} = T_p \widetilde{L}^D w^{\frac{DL}{L}}d^{D-1}x , \qquad d^D x = dx^0 \wedge \cdots \wedge dx^{D-1} , \tag{2.79}$$

$$D3 \quad : \quad \mathcal{L}_{magn} = -T_3 L^4 \omega_4 = -\frac{T_3 L^4}{4!}\epsilon^{\mu_1\mu_2\mu_3\mu_4}\omega_{\mu_1\mu_2\mu_3\mu_4}d^{D-1}x , \tag{2.80}$$

$$M5 \quad : \quad \mathcal{L}_{magn} = -\sqrt{T_5}L^3\omega_3 \wedge db = -\frac{\sqrt{T_5}L^3}{12}\epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}\omega_{\mu_1\mu_2\mu_3}\partial_{\mu_4}b_{\mu_5\mu_6}d^D x , \tag{2.81}$$

where the Lagrangians are expanded in positive powers of $\partial_\mu w^I/(uw)$, $f_{\mu\nu}/(uw)$ and $h_{\mu\nu\rho}/(uw)$ contracted with $\eta^{\mu\nu}$. In the IR limit, the electric WZ term cancel against the

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[12]For the M5 brane we use the formalism of [85, 86], with

$$\Phi(g,h) = \frac{T_5^{-1}}{12}g^{\mu_1\nu_1}g^{\mu_2\nu_2}g^{\mu_3\nu_3}h_{\mu_1\mu_2\mu_3}h_{\nu_1\nu_2\nu_3} \tag{2.76}$$

$$+ \frac{T_5^{-2}}{288}g^{\mu_1\nu_1}g^{\mu_2\nu_2}g^{\mu_3\nu_3}g^{\mu_4\nu_4}g^{\mu_5\nu_5}g^{\mu_6\nu_6}$$

$$\times (h_{\mu_1\mu_2\mu_3}h_{\nu_1\nu_2\nu_3}h_{\mu_4\mu_5\mu_6}h_{\nu_4\nu_5\nu_6} - 3h_{\mu_1\mu_2\mu_3}h_{\nu_2\nu_3\nu_4}h_{\mu_4\mu_5\mu_6}h_{\nu_5\nu_6\nu_1}) .$$

leading contribution from the kinetic term, while the magnetic terms, including the CS modification in $h = db + \sqrt{T_5}L^3\omega_3$, are fixed, which yields

$$
S_{M2} = -T_2 L^3 \int d^3x \frac{1}{4}\partial^\mu w_I \partial_\mu w_I ,
\tag{2.82}
$$

$$
S_{D3} = = -T_3 L^4 \int d^4x (\frac{1}{2}\partial^\mu w^I \partial_\mu w^I + \frac{1}{4Ng}f^{\mu\nu}f_{\mu\nu}) - T_3 L^4 \int \omega_4 ,
\tag{2.83}
$$

$$
S_{M5} = -T_5 L^6 \int d^6x (\frac{1}{16}\partial^\mu w_I \partial_\mu w_I + \frac{1}{24T_5 L^6}h^{\mu\nu\rho}h_{\mu\nu\rho})
$$

$$
-\sqrt{T_5}L^3 \int \omega_3 \wedge db , \qquad h = db + \sqrt{T_5}L^3\omega_3 .
\tag{2.84}
$$

Hence, the singular M2-brane is a free $\mathfrak{osp}(8|4)$ supersingleton [73], while the singular D3 and M5-branes are described by interacting $\mathfrak{psu}(2,2|4)$ and $\mathfrak{osp}(8^*|4)$ supersingletons, respectively. The purely bosonic sector of the interactions are given above, and thus governed by the scale-invariant scalar-field construct $\omega_{\widetilde{D}-1}$. Moreover, the singular M5-brane tensor-field equation is the non-linear self-duality condition $f_3 = \star f_3$ containing tensor-scalar interactions [85, 86]. The superconformal completions are given in [87], although they have not been worked out in detail in the singular limit.

3 THE TENSIONLESS LIMIT

In the previous section we have argued that the covariant phase space of tensile p-branes in anti-de Sitter spacetime contains partonic subspaces, represented semi-classically by singleton-like cusps on rotating p-branes in the limit $S \gg T_p L^{p+1} \gg 1$ and fixed $E - S$. Moreover, we have proposed an alternative description of the partonic degrees of freedom, interpolating smoothly from $TL^2 \gg 1$ to $TL^2 \ll 1$, in terms of decoupled massless open-p-brane excitations of $(D-2)$-branes at the boundary of anti-de Sitter spacetime. These considerations suggest a partonic spectrum given by symmetrized singletons (see (2.66)), raising the question whether these can be seen directly at the level of the tensionless limit of the closed p-brane in anti-de Sitter spacetime.

In this section we shall examine the limit $TL^2 \to 0$ of the bulk p-brane assuming that: 1) the *entire* p-brane has a faithful description in terms of the $(0+1)$-dimensional discretized Nambu-Goto action; 2) each fixed number, N, of discretized degrees of freedom, or fundamental partons, is an exact sector of the theory up to $1/N$ corrections[13]. We note that the partons carry space-time energy-momentum and spin but no other "internal" quantum numbers, as opposed to Thorn's colored string bits [88] (see also [20]).

We stress that in taking the tensionless limit at the level of the discretized action – where it is a weak coupling limit – we will end up with a model based on a Lagrangian quite different from the $(p+1)$-dimensional p-brane Lagrangian – where the limit is of course a strong coupling limit. The two models should therefore be compared at the level of the covariant phase space. To be more precise, we shall demonstrate a correspondence

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[13]This treatment harmonizes with that of the membrane in flat spacetime, with the crucial difference that the partonic spectrum in anti-de Sitter spacetime is discrete, with a well-defined interpretation in terms of space-time one-particle states.

between the partonic subspace of the covariant phase space of the tensionful brane and the fundamental *quantum* states of the discretized tensionless model.

Our two main findings are: first, if μ denotes the inverse lattice spacing, the proper tensionless limit is given by

$$L\mu \to 0 \; , \qquad T_p \mu^{-p-1} \to 0 \; , \tag{3.1}$$

resulting in an $\mathfrak{sp}(2N)$-gauged sigma model (see Section 3.2) on N copies of the Dirac hypercone. Second, the model has a well-defined continuum limit in $D = 7$ giving rise to a topological closed string (see Section 3.8) containing a singleton spin field. We also provide arguments why both systems reproduce the symmetrized-singleton spectrum (2.66).

A subtlety presents itself in that the spatial directions of the p-brane become blurred in the tensionless limit. Instead it makes sense to expect that the p-dependence enters via tensile deformations. Indeed, in the previous section we found that the cusps have p-dependent quantum properties, such that those on strings and membranes carry the same number of quantum degrees as actual singletons, while those on branes with $p > 2$ carry additional quantum degrees of freedom. The view that cusps on strings and membranes are deformed singletons, while those on higher p-branes are more complex, was lended further support by the fact that open p-branes ending at the branes at the boundary of anti-de Sitter spacetime can be made maximally supersymmetric maximally supersymmetric in setups with strings and membranes. Here we shall provide further evidence for the special status of $p = 1$ and $p = 2$, by showing that a simple tensile deformation that preserves S_N-invariance indeed lead to a dispersion relation between E and S of the same type as for the tensile membrane.

Moreover, we shall find that the singleton wave-functions involve subtleties and actual $\mathfrak{sp}(2N)$-anomalies that are remedied quantizing in phase space (see Sections 3.5 and 3.7), in turn facilitating the above continuum limit as well as the treatment in Section 4 of the massless two-parton system giving rise to Vasiliev's equations.

Before we turn to the discretization and continuum limit, we shall first discuss the issue of inequivalent massless limits of an ordinary point-particle in anti-de Sitter spacetime, focusing on the singular limit corresponding to (3.1), leading to the vector-oscillator realization of the scalar singleton. The different massless limits are discussed in Section 3.1.1, while the vector-oscillator realization of the singleton is described in Sections 3.1.2 and 3.1.3.

3.1   DIRAC'S HYPERCONE AND THE SINGLETON

3.1.1   PHASES OF THE $\mathfrak{sp}(2)$-GAUGED SIGMA MODEL

To describe the point particle in anti-de Sitter spacetime, it is most natural to start from the formulation in the ambient $2(D+1)$-dimensional phase space $\mathcal{Z}$ based on the first-order action

$$S = \int (\frac{1}{4} Y^{Ai} DY_{Ai} - \frac{1}{2}\Lambda^{ij} V_{ij}) \; , \tag{3.2}$$

with $Y^{Ai} \equiv \sqrt{2}(X^A, P^A)$, $DY^{Ai} \equiv dY^{Ai} + \Lambda^{ij}Y_j^A$, where Λ^{ij} is an $\mathfrak{sp}(2)$ gauge field, and V_{ij} a fixed function of τ. The classical field equations read

$$DY^{Ai} = 0 , \qquad K_{ij} + V_{ij} = 0 , \tag{3.3}$$

where K_{ij} is the $\mathfrak{sp}(2)$ generator

$$K_{ij} \equiv \frac{1}{2}Y_i^A Y_{jA} . \tag{3.4}$$

As a result, the angular momentum

$$M_{AB} = \frac{1}{2}Y_A^i Y_{Bi} \tag{3.5}$$

obeys the mass-shell condition

$$\frac{1}{2}M^{AB}M_{AB} = \frac{1}{2}V^{ij}V_{ij} . \tag{3.6}$$

With these normalizations, the underlying ungauged first-order system $\frac{1}{4}\int Y^{Ai}dY_{Ai}$ has Dirac bracket $\{Y_i^A, Y_j^B\}_D = 2\eta^{AB}\epsilon_{ij}$. For the choice

$$V_{ij} = \begin{pmatrix} M^2 & 0 \\ 0 & L^2 \end{pmatrix} . \tag{3.7}$$

the first-order equations of motion are equivalent to those of the point-particle on $(AdS_D)_L$ with mass M, described by the configuration space action

$$S[X^A] = \frac{1}{2}\int d\tau(e^{-1}\dot{X}^2 - eM^2 + \lambda(X^2 + L^2)) , \tag{3.8}$$

where e and λ are Lagrange multipliers.

For $(M, L) \neq (0, 0)$, the constraint (3.3) breaks $\mathfrak{sp}(2)$ to the $\mathbb{Z}_2$ acting as

$$\mathbb{Z}_2 : \quad M \leftrightarrow L \quad \Leftrightarrow X^A \leftrightarrow -P^A . \tag{3.9}$$

Already at the classical level, it is clear that the masslessness condition $ML = 0$ admits two branches:

i) the *massless particle*

$$M = 0 , \quad L > 0 \quad \leftrightarrow \quad M > 0 , \quad L = 0 , \tag{3.10}$$

moving along straight lines in the $(D+1)$-dimensional ambient spacetime intersecting $(AdS_D)_L$ defined by the hypersurface $X^2 + L^2 = 0$;

ii) the *singleton*

$$M = L = 0 , \tag{3.11}$$

living on the D-dimensional *Dirac hypercone*

$$X^2 = 0 , \qquad X^A \sim -X^A . \tag{3.12}$$

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Removing the apex gives a smooth manifold with two boundaries which is homotopic to a cylinder and has an $\mathfrak{so}(D-1,2)$-invariant $(D-1)$-bein e^a obeying

$$e^a(\partial_R) = 0 , \qquad (3.13)$$

where ∂_R is a the vector field pointing in the null direction of the cone. We shall propose a holographic interpretation of this geometry in Section 4.7. The corresponding singular line-element is defined by

$$ds^2 = e^a \otimes e^b \eta_{ab} . \qquad (3.14)$$

Splitting

$$X^A = (\overbrace{X^{0'},\underbrace{X^0,X^1,\ldots,X^{D-1}}_{X^a}}^{X^\alpha} \overbrace{\phantom{X^{D-1}}}^{X^r}) = R\widehat{X}^A , \quad \widehat{X}^\alpha\widehat{X}_\alpha = \widehat{X}^r\widehat{X}_r = 1 , \qquad (3.15)$$

yields the following parameterization of the singular line-element[14]

$$ds^2 = R^2(-d\hat{t}^2 + d\widehat{\Omega}_{D-2}^2) , \qquad R > 0 , \qquad (3.16)$$

where we note the absence of a dR^2 term. The singleton can perform many types of classical motion depending on the $\mathfrak{sp}(2)$ gauge choice [53]. In particular, constant $\omega^2 = \frac{1}{2}\Lambda_\tau^{ij}\Lambda_{\tau,ij}$ gives rise to five types of *singletonic motion*:

$\omega^2 < 0$: hyperbolic curves turning at a finite distance from the apex;

$\omega^2 = 0$: massless trajectories in the form of straight lines passing through the apex;

$\Lambda_{ij} = 0$: vacuum expectation values obeying $\dot{Y}^{iA} = 0$;

$\omega^2 > 0$: pulsations along finite straight intervals passing through the apex;

$\omega^2 > 0$: rotating trajectories forming closed loops in the time-like plane.

Upon quantization one introduces oscillators y^{iA} associated to the (constant) modes solving the classical equation of motion for Y^{iA}. The resulting non-commutative structure of the ambient phase space $\mathcal{Z}$ is given by

$$y_i^A \star y_j^B = y_i^A y_j^B + i\epsilon_{ij}\eta^{AB} , \qquad y_A^i = (y_A^i)^\dagger , \qquad (3.17)$$

where the $\star$ and juxtaposition denote the non-commutative and Weyl-ordered products, respectively, giving rise to the associative algebra $\mathcal{W}[\mathcal{Z}]$ based on the Moyal $\star$-product formula

$$f_1(y) \star f_2(y) = \int_{\mathcal{Z}\times\mathcal{Z}} \frac{d^{2(D+1)}S d^{2(D+1)}T}{(2\pi)^{2(D+1)}} e^{iT^{Ai}S_{Ai}} f_1(y_i^A + S_i^A)f_2(y_j^B + T_j^B) . \qquad (3.18)$$

---

[14]This form can also be obtained starting from the global non-singular coordinates $ds^2 = -\cosh^2\frac{r}{L}dt^2 + dr^2 + L^2\sinh^2\frac{r}{L}d\Omega_{D-2}^2$, via the reparameterization $r = \epsilon\widetilde{r} - L\log\epsilon$ followed by sending $\epsilon \to 0$ keeping $\widetilde{r}$, $\widetilde{L} = L/\epsilon$ and $R = \frac{\widetilde{L}}{2}e^{\frac{\widetilde{r}}{\widetilde{L}}} = \frac{L}{2}e^{\frac{\widetilde{r}}{L}}$ fixed. In Poincaré coordinates, $ds^2 = L^2(u^2dx^2 + \frac{du^2}{u^2})$, the singular limit can taken by keeping $\widetilde{u}\widetilde{L} = uL$ fixed, resulting in $ds^2 = \widetilde{u}^2\widetilde{L}^2dx^2$.

We note that the hermitian conjugation acts as

$$(f(y))^\dagger \equiv \bar{f}(y^\dagger) = \bar{f}(y) , \qquad (f \star g)^\dagger = g^\dagger \star f^\dagger . \tag{3.19}$$

The $\mathfrak{so}(D-1,2)$ and $\mathfrak{sp}(2)$ generators

$$M_{AB} = \frac{1}{2} y^i_{[A} \star y_{B]i} = \frac{1}{2} y^i_A y_{Bi} , \qquad (M_{AB})^\dagger = M_{AB} , \tag{3.20}$$

$$K_{ij} = \frac{1}{2} y^A_{(i} \star y_{j)A} = \frac{1}{2} y^A_i y_{jA} , \qquad (K_{ij})^\dagger = K_{ij} , \tag{3.21}$$

obey the commutation rules[15]

$$[M_{AB}, M_{CD}]_\star = 4i\eta_{[B|[C} M_{A]|D]} , \tag{3.22}$$

$$[K_{ij}, K_{kl}]_\star = 4i\epsilon_{(i|(k} K_{l)|j)} . \tag{3.23}$$

This single-oscillator realization leads to $\mathfrak{so}(D-1,2)$ representations carrying weights restricted by

$$M_{AB} \star M_{CD} \eta^{BD} = K^{ij} \star L_{ij,AC} - \frac{D-3}{2} \eta_{AC} - \frac{i(D-1)}{2} M_{AC} , \tag{3.24}$$

where $L^{AB}_{ij} = \frac{1}{2} y^A_{(i} y^B_{j)}$. Taking further traces yields

$$C_2[\mathfrak{so}(D-1,2)] = -4C_2[\mathfrak{sp}(2)] - \frac{1}{4}(D+1)(D-3) , \tag{3.25}$$

$$C_2[\mathfrak{so}(D-1)] = K^I \star L_{I,rr} + \frac{1}{2}(D-1)\left(E - \frac{D-3}{2}\right) , \tag{3.26}$$

$$L^+_r L^+_r = K^I \star (L_{I,00} + 2iL_{I,0} + L_I) , \tag{3.27}$$

where $L_{I,0} = v^A L_{I,0A}$, $L_I = v^A v^B L_{I,AB}$, and $C_2[\mathfrak{sp}(2)] = K^I \star K_I = -\frac{1}{8} K^{ij} \star K_{ij}$.

The massive-particle states are defined by the Casimir constraint

$$(\tfrac{1}{2} K^{ij} \star K_{ij} - M^2 L^2) \star |\Psi\rangle = 0 , \qquad ML > 0 , \tag{3.28}$$

which gives the AdS mass-shell condition, and one additional constraint that is linear in the $\mathfrak{sp}(2)$ generators, which can be taken to be either the embedding condition

$$(X^2 + L^2)|\Psi\rangle = 0 , \tag{3.29}$$

or, equivalently, via the $\mathbb{Z}_2$-symmetry (3.9), the "ambient" mass-shell condition

$$(P^2 + M^2)|\Psi\rangle = 0 . \tag{3.30}$$

These yield the $\mathfrak{so}(D-1,2)$ weight space $\mathfrak{D}(E_0, (0))$ with lowest energy $E_0 = \frac{D-1}{2} + \sqrt{1 + (ML)^2}$. The embedding condition (3.29) yields the wave-function

$$\Psi(X^A) = \delta(X^2 + L^2)\Phi(X^A) , \tag{3.31}$$

---

[15]For $D = 4$ our conventions give $[M_{12}, M_{23}] = iM_{13}$, that differ by a sign from the standard ones for $\mathfrak{so}(3)$. The canonical $\mathfrak{so}(2,1)$ generators are given by $K_I = -\frac{1}{4}(\sigma_I)^{ij} K_{ij}$, obeying $[K_I, K_J]_\star = i\epsilon_{IJK} K^L$, where $(\sigma^I)_{ij}$ are real and symmetric van der Waerden symbols defined by $(\sigma_I)_i{}^k (\sigma_J)_{kj} = \eta_{IJ}\epsilon_{ij} + \epsilon_{IJK}(\sigma^K)_{ij}$ with $\epsilon^{ij}\epsilon_{kj} = \delta^i_k$, $\eta_{IJ} = \text{diag}(+ + -)$ and $\epsilon^{IJK}\epsilon_{MNP} = -3!\delta^{IJK}_{MNP}$.

with $\Phi(X^A)$ obeying the massive Klein-Gordon equation, giving rise to a harmonic expansion with mode functions in $\mathfrak{D}(E_0, (0))$.

There are thus two *inequivalent* ways of sending $M \to 0$ at fixed $L > 0$: using (3.29), which leads to

$$(X^2 + L^2)|\Psi\rangle = 0 , \qquad K^{ij} \star K_{ij}|\Psi\rangle = 0 , \tag{3.32}$$

describing $\mathfrak{D}(E_0, (0))$ with $E_0 = \frac{D+1}{2}$ constituting the *massless scalar field on* $(AdS_D)_L$; or, using (3.30), which yields

$$P^2|\Psi\rangle = 0 , \qquad K^{ij} \star K_{ij}|\Psi\rangle = -\frac{1}{2}\left(\{X^A, P_A\}_\star + 4i\right) \star \{X^A, P_A\}_\star|\Psi\rangle = 0 , \tag{3.33}$$

in turn $\mathbb{Z}_2$-equivalent to a wave function on the Dirac hypercone, given by

$$\Psi(X) = \delta(X^2)\left(R^{-\Delta_-}\varphi_-(\widehat{X}) + R^{-\Delta_+}\varphi_+(\widehat{X})\right) , \tag{3.34}$$

with $\Delta_- = \epsilon_0 = \frac{D-3}{2}$ and $\Delta_+ = E_0 = \frac{D+1}{2}$, and where we use the coordinates defined in (3.16). Here $\varphi_\pm(x)$ are off-shell scalar fields living on the conifold $S^1 \times S^{D-2}$. Their harmonic expansions yield the Verma modules $V(E_0, (0)) \simeq \mathfrak{D}(E_0, (0))$, *i.e.* the D-dimensional massless scalar, and $V(\epsilon_0, (0))$, which contains the ideal

$$\mathcal{N}(\epsilon_0, (0)) \simeq V(E_0, (0)) . \tag{3.35}$$

Its elimination leaves the scalar singleton

$$\mathfrak{D}(\epsilon_0, (0)) = V(\epsilon_0, (0))/\mathcal{N}(\epsilon_0, (0)) , \tag{3.36}$$

containing the on-shell modes of the conformally coupled scalar on the conifold,

$$\left(\widehat{\nabla}^2_{S^1 \times S^{D-2}} - \frac{(D-3)^2}{4}\right)\varphi_-(\widehat{X}) = 0 . \tag{3.37}$$

This mass-shell condition is $\mathbb{Z}_2$-equivalent to augmenting (3.33) with $X^2|\Psi\rangle = 0$, as can be seen from (3.27), implying the strong $\mathfrak{sp}(2)$-invariance condition

$$K_{ij}|\Psi\rangle = 0 , \tag{3.38}$$

which thus singles out the singleton in the singular wave function (3.34).

3.1.2 SOME GEOMETRIC ASPECTS OF THE SINGLETON

The constraint (3.38), which corresponds to the limit (3.11), describes the "unbroken phase" of the $\mathfrak{sp}(2)$-gauged sigma model, with action

$$S = \frac{1}{4}\int Y^{Ai}DY_{Ai} . \tag{3.39}$$

The large gauge transformation

$$\rho : Y_i^A \mapsto -Y_i^A \tag{3.40}$$

descends to a reflection of the hypercone through the apex preserving the time orientation,

$$\rho \; : \quad (R, \widehat{X}) \mapsto (-R, \widehat{X}) \; . \tag{3.41}$$

The corresponding transformation of the singleton states read,

$$\rho |\Psi\rangle \;=\; (-1)^{\epsilon_0} |\Psi\rangle \; , \tag{3.42}$$

that is, the singleton exhibits a global anomaly unless $\epsilon_0 = 0 \bmod 2$, *i.e.* $D = 3 \bmod 4$.

Two other discrete maps are the combined space-time time reversal and parity transformation π, defined in (A.4) and (A.19), and the related τ-map defined in (A.3). Thus, if we let $|\alpha\rangle \equiv e^{i\pi\alpha/2} |\alpha + \epsilon_0, (\alpha)\rangle$ and $|\widetilde{\alpha}\rangle \equiv e^{i\pi\alpha/2} |-\alpha - \epsilon_0, (\alpha)\rangle$ $(\alpha = 0, 1, 2, \dots)$ denote real bases of the singleton $\mathfrak{D}$ and the anti-singleton $\widetilde{\mathfrak{D}}$, respectively, and $\langle \alpha| \equiv (|\alpha\rangle)^\dagger$ and $\langle \widetilde{\alpha}| \equiv (|\widetilde{\alpha}\rangle)^\dagger$ denote the corresponding bases of the dual spaces $\mathfrak{D}^\star$ and $\widetilde{\mathfrak{D}}^\star$, with normalization chosen to be

$$\langle \alpha | \beta \rangle \;=\; \langle \widetilde{\alpha} | \widetilde{\beta} \rangle \;=\; \delta_{\alpha\beta} \; , \tag{3.43}$$

then

$$\pi(|\alpha\rangle) \;=\; |\widetilde{\alpha}\rangle \; , \qquad \pi(\langle\alpha|) \;=\; \langle\widetilde{\alpha}| \; . \tag{3.44}$$

Furthermore, it can be seen from $\pi(L_r^\pm) = L_r^\mp$, $(L_r^\pm)^\dagger = L_r^\mp$ and $\tau(L_r^\pm) = -L_r^\pm$ that

$$\tau(|\alpha\rangle) \;=\; \langle\widetilde{\alpha}| \; , \qquad \tau(\langle\alpha|) \;=\; |\widetilde{\alpha}\rangle \; . \tag{3.45}$$

The π-map lifts to a parity transformation acting in $\mathcal{W}[\mathcal{Z}]$ as

$$\pi(f(y_i^a, y_i)) \;=\; f(y_i^a, -y_i) \; , \qquad y_i \;\equiv\; v_A y_i^A \; , \tag{3.46}$$

inducing a reversal in the orientation of a singleton world-line, as drawn in Fig. 3. The τ-map, which is a linear anti-automorphism, lifts to a map acting in $\mathcal{W}[\mathcal{Z}]$ as

$$\tau(f(y)) \;=\; f(iy) \; , \qquad \tau(f \star g) \;=\; \tau(g) \star \tau(f) \; , \tag{3.47}$$

which we also identify as a reversal of the orientation of the worldline. Thus, this geometric transformation can be represented either as an automorphism π, which separately exchanges incoming singletons and anti-singletons and out-going dittos, or an anti-automorphism τ, which exchanges incoming singletons with out-going anti-singletons and vice versa.

The singleton state space,

$$\mathcal{H} \;=\; \{|\Psi\rangle \; : \; K_{ij}|\Psi\rangle = 0\} \; , \tag{3.48}$$

is characterized by the Casimir relations (3.25) and (3.26), which imply

$$C_2[\mathfrak{so}(D-1,2)|\mathcal{H}] \;=\; -\tfrac{1}{4}(D+1)(D-3) \; , \tag{3.49}$$

$$C_2[\mathfrak{so}(D-1)|\mathcal{H}] \;=\; \frac{1}{2}(D-1)\Big(E - \frac{D-3}{2}\Big) \; . \tag{3.50}$$

These equations have two roots, given by

$$E_0 \;=\; \epsilon_0 \; , \qquad C_2[\mathfrak{so}(D-1)|S_0] \;=\; 0 \; , \tag{3.51}$$

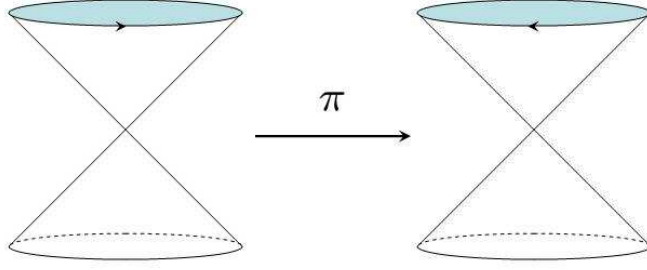$$E_0 \;=\; 1 \; , \qquad C_2[\mathfrak{so}(D-1)|S_0] \;=\; \frac{1}{4}(D-1)(5-D) \; . \tag{3.52}$$

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Figure 3: *Parity and time-reversal in ambient spacetime*: The parity transformation $v_A x_i^A \to -v_A x_i^A$ reverses the orientation of the worldline of a singleton rotating in ambient space.

The first root obeys the unitarity bound for $D \geq 3$. For $D = 3$ it corresponds to the trivial representation, while for $D \geq 4$ it leads to the *extended scalar singleton in $D \geq 4$*:

$$D \geq 4 : \qquad \mathfrak{D}_0 \equiv \mathfrak{D}(\epsilon_0, (0)) \oplus \widetilde{\mathfrak{D}}(-\epsilon_0, (0)) \ , \tag{3.53}$$

which is irreducible under $(\mathfrak{so}(D-1,2), \pi)$.

The second root (3.52) obeys the unitary bound only for $D = 3, 4, 5$. For $D = 5$ it coincides with (3.51). For $D = 4$, it corresponds to the spinor singleton $\mathfrak{D}(1, 1/2)$ of $\mathfrak{so}(3,2)$, which cannot be realized using vector oscillators[16]. For $D = 3$, the second root corresponds to a quasi-ground state of $\mathfrak{so}(2,2)$ with energy $E_0 = |S_0| = 1$, that combines with the trivial representation into the indecomposable *extended 3D scalar singleton*

$$D = 3 : \qquad \mathfrak{D}_0 = \mathfrak{D}(0,0) \oplus \sum_{\lambda = \pm 1} \left(\mathfrak{D}'(1, \lambda) \oplus \widetilde{\mathfrak{D}}'(-1, \lambda) \right) \ , \tag{3.54}$$

where $\mathfrak{D}'(\sigma, \lambda) = \{ |\sigma, \lambda; E, S\rangle : E = \lambda S = \sigma, 2\sigma, \dots \}$, and $L_{\lambda'}^{\pm} |\sigma, \lambda; E, S\rangle = c_{\pm, \lambda'; \sigma, \lambda; E, S} \times |\sigma, \lambda; E \pm 1, S + \lambda'\rangle$ with $c_{\pm, \lambda'; E, S; \sigma, \lambda} = 0$ iff $E = S = 0$ or $\lambda \lambda' = \pm 1$ (the Casimir formula applies since $L_r^{\pm} L_r^{\mp} |\pm 1, \pm 1\rangle = 0$). As a result, the *extended chiral 3D scalar singletons*:

$$\mathfrak{D}_0^{\pm} = \mathfrak{D}(0,0) \oplus \mathfrak{D}'(1, \pm 1) \oplus \widetilde{\mathfrak{D}}'(-1, \pm 1) \ . \tag{3.55}$$

are irreducible under $\{\mathfrak{so}(2,2), \pi\}$. The special character of $D = 3$ stems from the fact that the conformally coupled scalar field on $S^1 \times S^1$ has vanishing critical mass, which yields an unpaired zero mode corresponding to $\mathfrak{D}(0,0)$.

The singleton wave-function,

$$\Psi(X) = R^{-\Delta_-} \delta(X^2) \varphi_-(\widehat{X}) \ , \tag{3.56}$$

---

[16]The spinor-singleton root ultimately does give rise to a physical sector of the higher-spin gauge theory [38].

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has divergent norm, unlike ordinary particles that have normalizable wave-functions. Since the spectrum of the anti-de Sitter energy operator acting in a lowest weight space $\mathfrak{D}(E_0; S_0)$ is discrete, ordinary particles are quantized as if they were in a box, whereby wave-functions normalized to Kronecker δ-functions couple to well-defined second-quantized creation and annihilation operators. The wave-functions (3.56) do not lend themselves to this interpretation.

Let us therefore digress more into the details of the nature of the singleton wave-functions following [89, 29].

3.1.3 DIGRESSION: SINGLETON GROUND STATES

Instead of using the wave-function representation, it is convenient to work in the generalized Fock-space representation [89, 29] of the oscillators

$$w^A = \frac{1}{2}(y_1^A + iy_2^A) , \qquad \bar{w}^A = (w^A)^\dagger , \qquad [w^A, \bar{w}^B] = \eta^{AB} . \qquad (3.57)$$

The compact Fock-space vacuum defined by $(r = 1, \cdots, D-1, \alpha = 0, 0')$

$$w_r|0\rangle = \bar{w}_\alpha|0\rangle = 0 , \qquad (3.58)$$

yields the following normal-ordered expressions for the $\mathfrak{so}(D-1, 2)$ generators $M_{AB} = 2i\bar{w}_{[A}w_{B]}$,

$$E = u^\dagger \star u - v^\dagger \star v , \qquad M_{rs} = 2ia^\dagger_{[r} \star a_{s]} , \qquad (3.59)$$

$$L_r^- = \sqrt{2}\left(ua_r + v^\dagger a_r^\dagger\right) , \qquad L_r^+ = \sqrt{2}\left(va_r + u^\dagger a_r^\dagger\right) , \qquad (3.60)$$

where $a_r^\dagger \equiv i\bar{w}_r$ necessarily create states with integer $\mathfrak{so}(D-1)$ weights, $(s) \equiv (s0\ldots0)$, $s = 0, 1, \ldots$, and $u^\dagger \equiv (w_0 - iw_{0'})/\sqrt{2}$ and $v^\dagger \equiv (w_0 + iw_{0'})/\sqrt{2}$ are oscillators with integer $\mathfrak{so}(2)_E$-helicity, that can be used to create states with integer as well as half-integer energies. Thus, in view of (3.51) and (3.52), the ordinary Fock space

$$\mathcal{F} = \left\{ (u^\dagger)^m (v^\dagger)^n (a_r^\dagger a_r^\dagger)^p a_{\{r_1}^\dagger \cdots a_{r_n\}}^\dagger |0\rangle , \quad m, n, p = 0, 1, \ldots \right\} , \qquad (3.61)$$

suffices in odd dimensions, while even dimensions require a representation of u and v in a non-standard generalized Fock space, taken to be

$$\mathcal{F}' = \mathcal{F}_{1/2,0} \oplus \mathcal{F}_{0,1/2} , \qquad (3.62)$$

where

$$\mathcal{F}_{1/2,0} = \left\{ (u^\dagger)^{m+1/2} (v^\dagger)^n (a_r^\dagger a_r^\dagger)^p a_{\{r_1}^\dagger \cdots a_{r_n\}}^\dagger |0\rangle , \quad m \in \mathbb{Z}, \ n, p = 0, 1, \ldots \right\} , \qquad (3.63)$$

and the states $(u^\dagger)^{m+1/2}|0\rangle$ $(m \in \mathbb{Z})$ obey[17]

$$u(u^\dagger)^{m+1/2}|0\rangle = (m + 1/2)(u^\dagger)^{m-1/2}|0\rangle , \qquad (3.64)$$

$$\langle 0|u^{m+1/2}(u^\dagger)^{n+1/2}|0\rangle = \delta_{mn}\Gamma(m + 3/2)\langle 0|0\rangle . \qquad (3.65)$$

---

[17]In general, the state $(u^\dagger)^z|0\rangle = \Gamma(z+1) \oint_\gamma \frac{ds}{2\pi i} s^{-z-1} e^{u^\dagger s}|0\rangle$, where γ is the closed contour encircling $s = 0$ and $s = \infty$ with $\arg s = \pi + \arg u^\dagger$ where $\arg u^\dagger$ is determined by the eigenvalue of the dual coherent bra-state.

The introduction of the non-unitary sectors is necessary in order for the spaces to be an oscillator module. However, as we shall see, the physical states will belong to the unitary subsector.

The $\mathfrak{sp}(2)$ generators $K_{+-} = \frac{1}{2}\{w^A, \bar{w}_A\}_\star$, $K_{++} = (K_{--})^\dagger = w^A w_A$ read

$$K_{+-} = \frac{1}{2}(K_{11} + K_{22}) = a_r^\dagger \star a_r - u^\dagger \star u - v^\dagger \star v + \frac{D-3}{2} , \tag{3.66}$$

$$K_{++} = \frac{1}{2}(K_{11} - K_{22} + 2iK_{12}) = a_r a_r + 2u^\dagger v^\dagger , \tag{3.67}$$

$$K_{--} = \frac{1}{2}(K_{11} - K_{22} - 2iK_{12}) = a_r^\dagger a_r^\dagger + 2uv . \tag{3.68}$$

Therefore, a state $|E, (s)\rangle \in \mathcal{H}$ with AdS energy E and spin (s), which can be expanded as

$$|E, (s)\rangle = \Psi^{(s)}_{r_1...r_s} a_{r_1}^\dagger \cdots a_{r_s}^\dagger \Psi_{E,s}(u^\dagger, v^\dagger, a_r^\dagger a_r^\dagger)|0\rangle , \tag{3.69}$$

with traceless $\Psi^{(s)}_{r_1...r_s}$, is governed by the differential equations

$$(\partial_\xi - \partial_\eta - E)\Psi_{E,s} = 0 , \tag{3.70}$$
$$(2\partial_\zeta - \partial_\xi - \partial_\eta + \epsilon_0 + s)\Psi_{E,s} = 0 , \tag{3.71}$$
$$(2(\partial_\zeta^2 + (\epsilon_0 + s)\partial_\zeta) + e^{\xi+\eta+\zeta})\Psi_{E,s} = 0 , \tag{3.72}$$
$$(e^{\xi+\eta+\zeta} + 2\partial_\xi\partial_\eta)\Omega_{E,s} = 0 , \tag{3.73}$$

where

$$\xi = \log u^\dagger , \quad \eta = \log v^\dagger , \quad \zeta = \log a_r^\dagger a_r^\dagger . \tag{3.74}$$

Eqs. (3.70) and (3.71) have the solution

$$\Psi_{E,s} = e^{\frac{E}{2}(\xi-\eta) - \frac{\epsilon_0+s}{2}\zeta} f_E(x) , \qquad x \equiv e^{\xi+\eta+\zeta} = u^\dagger v^\dagger a_r^\dagger a_r^\dagger . \tag{3.75}$$

The remaining conditions (3.72) and (3.73) imply

$$\left((x\frac{d}{dx})^2 + \frac{1}{2}(x - \frac{E^2}{2})\right) f_E(x) = 0 , \qquad E^2 = (\epsilon_0 + s)^2 , \tag{3.76}$$

i.e. $f_E(x)$ obeys Bessel's differential equation in $z = \sqrt{2x}$ with index $|E|$.

To describe the solutions one introduces the contour integral

$$F_{\nu,\gamma}(z) = \int_\gamma \frac{ds}{2\pi i}(1 - s^2)^{\nu-1/2}e^{sz} , \tag{3.77}$$

which solves the related differential equation

$$\left(z - (2\nu + 1)\frac{d}{dz} - z\frac{d^2}{dz^2}\right)F_{\nu,\gamma}(z) = 0 , \tag{3.78}$$

provided that

$$\left[(1 - s^2)^{\nu+1/2}e^{sz}\right]_{\partial\gamma} = 0 , \tag{3.79}$$

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so that $F_{\nu,\gamma}(z)/z^\nu$ solves Bessel's differential equation in z with index ν. The contours with $\partial\gamma \in \{\pm 1, \pm\infty\}$ correspond to the Bessel and Neumann functions. It is also possible to take $\partial\gamma = \{-i\infty, i\infty\}$, by making use of the prescription

$$\delta(z) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} e^{sz+\epsilon s^2} , \tag{3.80}$$

where $\epsilon \to 0^+$. This yields the distribution

$$F_\nu(z) = \sum_{k=0}^{\infty} \binom{\nu - 1/2}{k}(-1)^k \delta^{(2k)}(z) , \tag{3.81}$$

that has a well-defined action on test functions $h(z)$ ($z \in \mathbb{R}$) with $h^{(k)}(0)$ falling off with k sufficiently fast, as to justify the geometric series expansion. Indeed, (3.81) solves (3.78) in the sense that

$$\int_{-\infty}^{\infty} dz \; h(z) \left(z - (2\nu + 1)\frac{d}{dz} - z\frac{d^2}{dz^2} \right) F_\nu(z) = 0 . \tag{3.82}$$

In summary, there are two solutions corresponding to states in the generalized Fock spaces, namely the function

$$f_E^{(1)}(x) \equiv \mathcal{N}_E \; J_{|E|}(\sqrt{2x}) , \tag{3.83}$$

which is analytic at the origin, and the distribution

$$f_E^{(2)}(x) = \mathcal{N}'_E (2x)^{\frac{|E|}{2}} F_{|E|}(\sqrt{2x}) . \tag{3.84}$$

The solution related to the Neumann function, which contains a logarithm, will not be considered here. The existence of both analytic and distributional wave-functions is analogous to the existence of analytic and distributional phase-space and spinor-space projectors found in [11].

The action of $L_r^\pm$ on the states $|E, (s)\rangle^{(q)}$ built from $f_E^{(q)}(x)$ ($q = 1, 2$) is given by $L_r^\pm |E, (s); q\rangle = c_{E,s} |E \pm 1, (s \pm \frac{E}{|E|}); q\rangle$. Since $s = |E| - \epsilon_0 \geq 0$, it follows that if $\epsilon_0 \geq 1/2$, i.e. $D \geq 4$, then

$$D \geq 4 : \quad |\Omega_\pm; q\rangle \equiv |\pm \epsilon_0, (0); q\rangle , \tag{3.85}$$

are singleton ground states, obeying $L_r^\mp |\Omega_\pm; q\rangle = 0$. Indeed, $L_r^- |\epsilon_0, (0); q\rangle$ has a wave-function proportional to $(2\partial_\xi\partial_\zeta + e^{\xi+\eta+\zeta})\Psi_{\epsilon_0,0}^{(q)}$, that in turn vanishes due to (3.70-3.73). For $D = 3$, the states $L_r^\pm |E, (s); q\rangle$ with $|E| = s \geq 1$ are non-vanishing, while $L_r^\pm |0, (0); q\rangle = 0$. Thus $|\pm 1, (1); q\rangle$, each of which decompose under $\mathfrak{so}(2)_S$ into helicity eigen-states $|\pm 1, \lambda; q\rangle$ with $\lambda = \pm 1$, are the quasi-ground states for the spaces $D'(1, \lambda)$ and $\widetilde{D}'(-1, \lambda)$ defined in (3.54).

In summary,

$$\mathcal{H} = \mathfrak{D}_0^{(1)} \oplus \mathfrak{D}_0^{(2)} , \tag{3.86}$$

where $\mathfrak{D}_0^{(q)}$ are extended singletons isomorphic to (3.53) and (3.54). The phase-space reflection (3.46) acts as $\pi(\Psi(u^\dagger, v^\dagger, a_r^\dagger)) = \Psi(v^\dagger, u^\dagger, a_r^\dagger)$. Thus, declaring $\pi|0\rangle = |0\rangle$, leads

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to that $\pi(|\Omega_\pm; q\rangle) = |\Omega_\mp; q\rangle$, so that $D_0^{(q)}$ are irreducible under $\{\mathfrak{so}(D-1,2), \pi\}$, in the sense of (A.19).

The explicit expressions for the ground states read [29]

$$|\Omega_\sigma; q\rangle = \mathcal{N}^{(q)} \left[(u^\dagger)^{(1+\sigma)/2} (v^\dagger)^{(1-\sigma)/2} \right]^{\epsilon_0} \int_{\gamma^{(q)}} ds\, (1-s^2)^{\epsilon_0 - 1/2}\, e^{s\sqrt{2u^\dagger v^\dagger a_r^\dagger a_r^\dagger} + \epsilon s^2} |0\rangle\,, \quad (3.87)$$

where $\gamma^{(1)} = [-1, 1]$ and $\gamma^{(2)} =] - i\infty, i\infty[$. The analytic ground states

$$|\Omega_\sigma; 1\rangle = \left[(u^\dagger)^{(1+\sigma)/2} (v^\dagger)^{(1-\sigma)/2} \right]^{\epsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} \frac{\Gamma(\frac{D-1}{2})}{\Gamma(n + \frac{D-1}{2})} (u^\dagger v^\dagger a_r^\dagger a_r^\dagger)^n |0\rangle\,, \quad (3.88)$$

are highly squeezed oscillator states, intersecting only the positively normed subspaces of $\mathcal{F}_{1/2,0}$ and $\mathcal{F}_{0,1/2}$. For $D = 3$, the analytic quasi-ground states are given by,

$$|\sigma, \lambda; 1\rangle = \lambda^r a_r^\dagger (u^\dagger)^{(1+\sigma)/2} (v^\dagger)^{(1-\sigma)/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n n!(n+1)!} (u^\dagger v^\dagger a_r^\dagger a_r^\dagger)^n |0\rangle\,. \quad (3.89)$$

The norms of the ground states have a divergent nature, $e.g.$

$$\||\Omega_\pm; 1\rangle\|^2 = \Gamma(\tfrac{D-1}{2}) \sum_{n=0}^{\infty} \langle 0|0\rangle\,. \quad (3.90)$$

Similarly, the inner products $\langle \Omega_\pm; 2|\Omega_\pm; 2\rangle$ and $\langle \Omega_\pm; 2|\Omega_\pm; 1\rangle$ have divergent integral representations.

We *conjecture that there exists a well-tempered linear combination of the analytical and the distributive ground states with finite norm,*

$$|\widehat{\Omega}_\pm\rangle = c_1 |\Omega_\pm; 1\rangle + c_2 |\Omega_\pm; 2\rangle\,, \quad (3.91)$$

where $c_{1,2}$ are finite constants and

$$\||\Omega_\pm\rangle\|^2 = 1\,, \quad (3.92)$$

corresponding to a proper normalizable singleton subspace of (3.86),

$$\widehat{\mathcal{H}} = \mathrm{diag}\left(c_1 \mathfrak{D}_0^{(1)} \oplus c_2 \mathfrak{D}_0^{(2)} \right) \simeq \mathfrak{D}_0\,. \quad (3.93)$$

3.1.4   Remark on the D-Dimensional Scalar

Out of curiosity, let us examine the weakly $\mathfrak{sp}(2)$-invariant spaces

$$\mathcal{H}_\pm = \left\{ |\Psi_\pm\rangle\ :\quad K_{\pm\pm} |\Psi_\pm\rangle = K_{\pm\mp} |\Psi_\pm\rangle = 0 \right\}\,, \quad (3.94)$$

which contain the singleton, $\mathcal{H}_\pm \supset \mathcal{H}$, that can be projected out by imposing either $K_{\mp\mp} |\Psi_\pm\rangle = 0$ or, *equivalently*, one of the lowest-weight conditions $L_r^- |\Psi_\pm\rangle = 0$ or $L_r^+ |\Psi_\pm\rangle = 0$. Hence, the only lowest-weight module contained in $\mathcal{H}_\pm$ is the singleton.

The spaces $\mathcal{H}_\pm$ also contain a D-dimensional scalar, residing in the manifestly Lorentz-covariant $\mathfrak{so}(D-1,2)$ module

$$V_\pm = \bigoplus_{n=0}^\infty P_{\{a_1} \star P_{a_n\}}|\Omega_\pm\rangle , \qquad K_{\pm\pm}|\Omega_\pm\rangle = K_{\pm\mp}|\Omega_\pm\rangle = M_{ab}|\Omega_\pm\rangle = 0 , \qquad (3.95)$$

which is not of lowest-weight type.

Using the Lorentz-covariant oscillator vacuum $(a = 0, \ldots, D-1)$

$$\alpha_a|\hat{0}\rangle = \beta|\hat{0}\rangle = 0 , \qquad (3.96)$$

where $\alpha_a \equiv w_a$ and $\beta \equiv \bar{w}_{0'}$, the conditions on $|\Omega_+\rangle = \Omega_+(\alpha^{a\dagger}\alpha_a^\dagger, \beta^\dagger)|\hat{0}\rangle$ become

$$(\alpha^a \alpha_a - (\beta^\dagger)^2) \, \Omega_+(\alpha^{a\dagger}\alpha_a^\dagger, \beta^\dagger)|\hat{0}\rangle = 0 , \qquad (3.97)$$

$$(\alpha^{a\dagger}\alpha_a - \beta^\dagger \beta + \frac{D-1}{2}) \, \Omega_+(\alpha^{a\dagger}\alpha_a^\dagger, \beta^\dagger)|\hat{0}\rangle = 0 , \qquad (3.98)$$

which are equivalent to $\Omega_+ = (\beta^\dagger)^{\frac{D-1}{2}} f_+(y)$, $y^2 = \alpha^{a\dagger}\alpha_a^\dagger(\beta^\dagger)^2$, with

$$\left(y \frac{d^2}{dy^2} + (D-1)\frac{d}{dy} - y \right) f_+(y) = 0 , \qquad (3.99)$$

i.e. $y^{-\nu} f_+(y)$ obeys Bessel's differential equation in y with index $\nu = \frac{D-2}{2}$ (that happens to coincide with the index of the Bessel functions related to the singular projector M defined in (3.147)). This implies that

$$P_a|\Omega_+\rangle = i(3-D)(\beta^\dagger)^{\frac{D+1}{2}} \alpha_a^\dagger (2y)^{-1} f_L'(y)|0_+\rangle . \qquad (3.100)$$

For $D = 3$, V_+ is trivial, and indeed $K_{--}|\Omega_+\rangle = 0$, so that $V_+ = \mathfrak{D}(0,0)$. For $D > 3$, V_+ is a non-trivial representation space for $\mathfrak{so}(D-1,2)$, with $C_2[\mathfrak{so}(D-1,2)|V_+]$ given by (3.49), since $C_2[\mathfrak{sp}(2)|V_+] = 0$.

3.2 $\mathfrak{sp}(2N)$-GAUGED BRANE-PARTON MODEL

In this section we examine the tensionless limit of the discretized p-brane in anti-de Sitter spacetime at the classical level. We are working covariantly in embedding space, *i.e.* we do not solve the gauge conditions on the embedding fields. As a result, we end up with a model with an enhanced gauge symmetry, restricting the classical partonic phase space substantially, while being less restrictive at the quantum level, leaving the expected partonic quantum states, as we shall discuss in Section 3.4.

To discretize the Nambu-Goto action (2.1) [90][18], it is convenient to follow the approach in which one introduces an auxiliary metric, $\gamma_{\alpha\beta}$, so that

$$S_p = -\frac{T_p}{2} \int d^{p+1}\sigma \sqrt{-\det g} \left(\gamma^{\alpha\beta} g_{\alpha\beta} - (p-1) + \Lambda(X^2 + L^2) \right) , \qquad (3.101)$$

---

[18]The discretization procedure can also be performed fully covariantly using the einbein-formulation given in [24].

and then integrating out the spatial metric γ_{rs} in the gauge $\gamma_{00} = -\det \gamma_{rs}$ and $\gamma_{0r} = 0$ ($\alpha \to (0, r)$, $r = 1, \ldots, p$), imposed using the $(p + 1)$-dimensional diffeomorphisms. After re-scaling $\sigma^r \to T_p^{-1/p} \sigma^r$, one finds

$$S_p = \frac{1}{2} \int d^{p+1}\sigma \left(\dot{X}^2 - T_p^2 \det{}'g + \Lambda(X^2 + L^2) \right) , \tag{3.102}$$

where $\det{}'g = \det g_{rs}$. The physical states obey the mass-shell condition and Gauss' law,

$$\dot{X}^2 + T_p^2 \det{}'g = 0 , \qquad \dot{X}^A \partial_r X_A = 0 , \tag{3.103}$$

and the second-class embedding constraints

$$X^2 + L^2 = 0 , \qquad \dot{X}^A X_A = 0 . \tag{3.104}$$

To discretize one replaces

$$X^A(\tau, \sigma^r) \to \left\{ X^A(\tau; \xi) \right\}_{\xi=1}^N , \tag{3.105}$$

where $X^A(\tau; \xi)$ are degrees of freedom, referred to as partons (see also [88]), living on N sites labelled by ξ. This requires a prescription for replacing derivatives with respect to σ^r by difference operators,

$$\frac{\partial X^A}{\partial \sigma^r} \to \mu^{-1/p} \sum_\eta w_{\xi,\eta} X^A(\eta) , \qquad \sum_\eta w_{\xi,\eta} = 0 , \tag{3.106}$$

where μ is a fixed mass parameter, and $w_{\xi\eta}$ are some weights, which give rise to interactions that are irrelevant in the limit $T_p \ll \mu^{p+1} \to 0$. Gauss' law now takes the form $\sum_\eta w_{\xi,\eta} \dot{X}^A(\xi) X_A(\eta) = 0$. Demanding closure of the constraint algebra in the limit $\mu L = 0$ [91] *and* $\mu^{-p-1} T_p = 0$ [22, 24, 92], we find that for *any* choice of weights the only possibility is the $\mathfrak{sp}(2N)$ constraint

$$\dot{X}^A(\xi) X_A(\eta) = 0 , \qquad \forall \, \xi, \eta . \tag{3.107}$$

Thus we find an $\mathfrak{sp}(2N)$-gauged sigma model, with the following phase-space action

$$S_N = \frac{1}{4} \int Y^{AI} DY_{AI} , \tag{3.108}$$

where I labels an $2N$-plet, and ($\xi = 1, \ldots, N$; $i = 1, 2$)

$$Y^{AI} = \{Y^{Ai}(\xi)\} , \qquad Y^{Ai}(\xi) = \sqrt{2} \left(X^A(\xi), P^A(\xi) \right) , \tag{3.109}$$

are the phase-space coordinates of the partons, and the $\mathfrak{sp}(2N)$-covariant derivative is defined by

$$DY^{AI} = dY^{AI} + \Lambda^{IJ} Y_J^A , \tag{3.110}$$

where Λ^{IJ} is a $(0 + 1)$-dimensional $\mathfrak{sp}(2N)$ gauge field. The symplectic indices are raised and lowered as $Y^{AI} = \Omega^{IJ} Y_J^A$ and $Y_I^A = Y^{AJ} \Omega_{JI}$ with $\Omega_{IJ} = \epsilon_{ij} \delta_{\xi\eta}$.

The $\mathfrak{sp}(2N)$ generators

$$T_{IJ} = \frac{1}{2} Y_I^A Y_{AJ} \tag{3.111}$$

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consist of N $\mathfrak{sp}(2)$ blocks $K_{ij}(\xi)$ along the diagonal containing the mass-shell conditions and geometric constraints of the separate partons, and off-diagonal generators $T_{i,j}(\xi,\eta) = Y_i^A(\xi)Y_{Aj}(\eta)$, $\xi \neq \eta$, comprising the discretized Gauss' law as well as additional $\mathcal{W}$-constraints of a continuum limit to be discussed in more detail below.

The large gauge transformations are generated by phase-space *reflections*

$$\rho_\xi : \ Y^{Ai}(\eta) \to (-1)^{\delta_{\xi\eta}} Y^{Ai}(\eta) \ , \tag{3.112}$$

and *permutations*

$$P_{\xi\eta} : \ Y^{Ai}(\xi) \to Y^{Ai}(P(\xi)) \ . \tag{3.113}$$

The latter are large discrete diffeomorphisms arising at the point $T_p = 0$, where thus the original p-dimensional nature of the worldvolume is lost.

The classical equations of motion,

$$T_{IJ} \ = \ 0 \ , \qquad \dot{Y}_I^A + \Lambda_I{}^J Y_J^A = 0 \ , \tag{3.114}$$

imply that the classical space-time angular momenta, defined by

$$M_{AB} = \frac{1}{2} Y_A^I Y_{IB} \ , \tag{3.115}$$

are light-like,

$$M_A{}^C M_{BC} \ = \ 0 \ , \tag{3.116}$$

where we note that

$$M_A{}^C M_{BC} \ = \ T^{IJ} L_{IJ,AB} \ , \qquad L_{IJ}{}^{,AB} \ = \ \frac{1}{2} Y_{(I}^A Y_{J)}^B \ . \tag{3.117}$$

This degeneracy is lifted in the quantum theory, which has a discrete spectrum, essentially due to the underlying $\mathfrak{so}(D-1,2)$ covariance.

The naive dimension of the space $\mathcal{M}_N$ of classical solutions is given by $2N(D+1) - 2 \times \frac{2N(2N+1)}{2} = 2N(D-2N)$, that becomes negative for large enough N. The actual dimension is, however, positive for all N. To describe the solutions one can fix the harmonic gauge

$$\Lambda^I{}_J \ = \ \mathbf{1}_{N \times N} \otimes \epsilon \ , \tag{3.118}$$

where $\epsilon = i\sigma^2$. The solutions to (3.114) are now given by

$$Y_1^A(\xi) \ = \ q^A(\xi) \cos \tau + p^A(\xi) \sin \tau \ , \tag{3.119}$$

where $q^A(\xi)$ and $p^A(\xi)$ are constants of motion obeying

$$q(\xi) \cdot q(\eta) = q(\xi) \cdot p(\eta) = p(\xi) \cdot p(\eta) = 0 \ , \qquad \forall \xi, \eta \ . \tag{3.120}$$

The space of classical solutions is obtained by factoring out the group G of residual gauge transformations preserving (3.118),

$$\mathcal{M}_N \ = \ \left\{ \left(q^A(\xi), p^A(\xi) \right) \right\} / G \ , \qquad G = SO(2N) \cap Sp(2N) \ . \tag{3.121}$$

A particularly simple class of solutions is thus given by

$$q(\xi) \;=\; q \;, \qquad p(\xi) \;=\; p \;, \qquad (q,p) \in \mathcal{M}_1 \;. \tag{3.122}$$

For $q \neq p$ these solutions are rotating clusters of partons, with

$$M_{AB}(N) \;=\; N M_{AB}(1) \;, \tag{3.123}$$

while $q = p$ leads to "pulsating" clusters, with

$$M_{AB}(N) \;=\; 0 \;. \tag{3.124}$$

We see that $\dim \mathcal{M}_N \geq \dim \mathcal{M}_1$ for all N, while $\dim \mathcal{M}_N \ll N \dim \mathcal{M}_1$ as N becomes large. Thus, the space of classical solutions to the discretized tensionless model is much smaller than the space of cusp solutions of the tensile branes. This puzzle is resolved, however, at the quantum level, where the gauge symmetry is much less restrictive, as we shall see in Section 3.4.

Before turning to this crucial issue, let us briefly address the universality of the mechanism by which the constraints of the singular geometry and the p-dimensional diffeomorphisms combine into an $\mathfrak{sp}(2N)$ gauge algebra. This suggests that different p-branes result from different tensile deformations. However, as described in Section 2.4, the cusps on p-branes with $p > 2$ give rise to additional degrees of freedom that cannot directly be related to singletons. Moreover, for $p = 1$ the cusps interact via a linear potential, suggesting that S_N-invariance is actually broken down to $\mathbb{Z}_N$-invariance, as discussed in Section 2.3. A natural resolution could be that tensile deformations lead to the string in case they preserve $\mathbb{Z}_N$-invariance and to the membrane in case they preserve S_N-invariance. Let us demonstrate the latter in a particular example.

3.3   A Tensile Membrany Deformation

A small tensile perturbation of the $\mathfrak{sp}(2N)$-gauged action that preserves the S_N-invariance is given by

$$S \;=\; \frac{\mu}{2} \int d\tau \sum_{\xi=1}^{N} \sum_{\eta \neq \xi} \left\{ \dot{X}^2(\xi) - k^2 \big(X(\xi) - X(\eta) \big)^2 + \Lambda(\xi) \big(X^2(\xi) + L^2 \big) \right\} \;, \tag{3.125}$$

with corresponding Virasoro-like constraints

$$\dot{X}^2(\xi) + k^2 \sum_{\eta \neq \xi} \big(X(\xi) - X(\eta) \big)^2 \;=\; 0 \;, \tag{3.126}$$

$$\sum_{\eta \neq \xi} \Big(X(\xi) \cdot \dot{X}(\eta) - X(\eta) \cdot \dot{X}(\xi) \Big) \;=\; 0 \;, \tag{3.127}$$

where $k \equiv T_p/\mu^{p+1} \ll 1$, the long-range difference operator corresponds to the behavior of the leading-order interactions between the cusps, and, in a self-consistent fashion, the $\mathfrak{sp}(2N)$ gauge symmetry has been partly gauged fixed and partly broken, leaving N Lagrange multiplies $\Lambda(\xi)$ that will be determined by the equations of motion.

The classical solution describing co-planar and unison rotation with mutual relative azimuthal angles $2\pi/N$ reads

$$X^{0'}(\xi) + iX^0(\xi) = L\cosh\frac{r_0}{L}e^{i\omega(r_0)\tau} , \qquad (3.128)$$

$$X^1(\xi) + iX^2(\xi) = L\sinh\frac{r_0}{L}e^{i\tilde{\omega}(r_0)\tau}e^{2\pi(\xi-1)/N} , \qquad (3.129)$$

$$X^A = 0 , \quad A = 3,\ldots,D-1 , \qquad (3.130)$$

where the equations of motion and constraints determine the angular velocities to be $\omega^2(r_0) = \tilde{\omega}^2(r_0) - 2k^2N = 4k^2N\sinh^2(r_0/L)$. The discrete symmetry of the configuration implies that $M_{AB}(\xi)$ is independent of ξ, resulting in total energy and spin given by

$$E_{\rm cl} = NL^2\omega(r_0)\cosh^2\frac{r_0}{L} , \qquad (3.131)$$

$$S_{\rm cl} = NL^2\tilde{\omega}(r_0)\sinh^2\frac{r_0}{L} . \qquad (3.132)$$

In the limit $r_0/L \ll 1$, the dispersion relation becomes $E_{\rm cl}^2 \sim N^{3/2}L^2kS_{\rm cl}$, reminiscent of that of a short string, *i.e.* a string in flat spacetime. On the other hand, for $r_0/L \gg 1$, the classical charges scale like $kL^2\exp(3r_0/L)$, and

$$E_{\rm cl} - S_{\rm cl} \sim (k^2S_{\rm cl})^{1/3} , \qquad k \ll 1 , \qquad (3.133)$$

in agreement with the near tensionless behavior of the folded rotating tensile membranes in AdS_D found in (2.46). This result is consistent with the expected S_N-invariance of the two-dimensional gas of cusps on the tensile membrane. Indirectly, this also lends suppert to the idea that tensile deformations that break S_N to $\mathbb{Z}_N$ should lead to stringy dispersion relations.

3.4 QUANTIZATION AND GLOBAL WAVE-FUNCTION ANOMALIES

As we found in Section 3.2, the phase space of the classical $\mathfrak{sp}(2N)$-gauged model is too restricted to match that of the cusps on the tensile brane. However, at the quantum level, the expectation values of the $\mathfrak{sp}(2N)$ generators vanish provided the N-parton states $|\Psi\rangle$ obey[19]

$$K_{ij}(\xi)|\Psi\rangle = 0 , \qquad \xi = 1,\ldots,N . \qquad (3.134)$$

This condition can be solved by tensoring N normalizable single-parton states belonging to $\widehat{\mathcal{H}} \simeq \mathfrak{D}_0$, *i.e.* the scalar singleton, as given in (3.93). Thus, the quantum state space is much richer than the classical phase space, *i.e.* the correspondence principle is not valid.

Turning to the invariance under large $Sp(2N)$ transformations, the gauging of the permutations (3.113), which amounts to imposing

$$P_{\xi\eta}|\Psi\rangle = |\Psi\rangle , \qquad (3.135)$$

---

[19]The N-oscillator analog of (3.24),

$$M_{AC} \star M_B{}^C = K^{IJ} \star L_{IJ,AB} - \frac{i}{2}(D-1)M_{AB} - \frac{N}{2}(D-2N-1)\eta_{AB} ,$$

shows that the strong $\mathfrak{sp}(2N)$-invariance condition $K_{IJ}|\Psi\rangle = 0$ is incompatible with the unitarity bounds, except for certain "sporadic" values of $N > 1$ and D, that we shall not consider here.

implies that the multi-parton spectrum is fully Bose symmetrized, *viz*

$$\mathcal{S}_{\text{partons}} = \bigoplus_{N=1}^{\infty} [\mathfrak{D}_0]_{\text{symm}}^{\otimes N} . \tag{3.136}$$

Thus, there is a correspondence between the quantized discretized tensionless model and the partonic region of the phase space of the tensile brane. We also note that the state space (3.136) is isomorphic to the the spectrum (2.66) of the singular conformal field theory, which, strictly speaking, also contains singletons and anti-singletons in $\mathfrak{D}_0$ realized as second-quantized creation and annihilation operators.

However, in view of (3.42), the gauging of the reflections (3.112) is, however, possible in $\mathcal{S}_{\text{partons}}$ only for

$$D = 3 \mod 4 , \tag{3.137}$$

while it leads to the unnatural truncation $N\epsilon_0 = 0 \mod 2$ for other values of D.

The global anomaly seems unnecessary in the massless sector, and moreover, it would be desirable with a more transparent formulation free from the subtleties associated with the wave-functions.

3.5   PHASE-SPACE APPROACH

The phase-space formulation of quantum mechanics is based on a deformed product of phase-space functions, denoted by $\star$; a set of functions with well-defined $\star$-composition, referred to as operators; and, a trace operation Tr analogous to phase-space integration. There is no *a priori* reference to states, or any probabilistic interpretation, so the quantum theory presents a smooth deviation away from the classical Liouville equation [93, 94, 95, 96].

Field-theoretically, the approach is based on phase-space correlators with boundary conditions of various complexity. The basic formulation uses symmetric conditions at $\tau \to \pm\infty$, *viz.*

$$\langle \mathcal{O} \rangle_y \equiv \int_{Y(\pm\infty)=y} \mathcal{D}Y \, \mathcal{O}[Y] e^{\frac{i}{\hbar} S[Y]} , \tag{3.138}$$

where Y denote the phase-space coordinate, y a point in phase space, so that the standard trace becomes

$$Tr\mathcal{O} = \int [dy] \langle \mathcal{O} \rangle_y . \tag{3.139}$$

The sigma-model action S is assumed to be diffeomorphism invariant, which in general may require additional gauge fields, so that expectation values of ultra-local operators $\mathcal{O}_f = f(Y(\tau))$, where f are phase-space functions, are given by $\langle \mathcal{O}_f \rangle_y = f(y)$, independently of τ. The $\star$-product is then induced from expectation values

$$\langle \mathcal{O}_{f_1} \mathcal{O}_{f_2} \rangle_y = \epsilon(\tau_1 - \tau_2)(f_1 \star f_2)(y) + \epsilon(\tau_2 - \tau_1)(f_2 \star f_1)(y) . \tag{3.140}$$

We shall use the notation $Tr\mathcal{O}_f = Trf$ and $Tr[\mathcal{O}_{f_1}\mathcal{O}_{f_2}] = Tr[f_1 \star f_2]$.

In the phase-space formulation, external multi-singleton states $\Psi_{1...P} = [|\alpha_1\rangle \otimes \cdots \otimes |\alpha_P\rangle]_{\mathrm{symm}}$ are represented by vertices

$$\mathcal{V} \in \bigoplus_{\substack{P_1 + P_2 = P \\ P_{1,2} > 0}} \left[(\mathfrak{D}^{\otimes P_1} \otimes \widetilde{\mathfrak{D}}^{\star \otimes P_2}) \oplus (\widetilde{\mathfrak{D}}^{\otimes P_1} \otimes \mathfrak{D}^{\star \otimes P_2}) \right]_{\mathrm{symm}} , \qquad (3.141)$$

describing processes in which P_2 singletons are emitted while P_1 singletons are absorbed, whereby the total number of internal singletons is changed by $P_1 - P_2$. The vertices act in the space $\mathcal{S}_{\mathrm{partons}}$ given in (3.136), which we may view as a generalized Chan-Paton factor representing an internal singleton cluster.

The two-singleton vertices act invariantly (and in fact irreducibly) on each term in the generalized Chan-Paton factor. Their action on the single singleton corresponds to $\mathfrak{sp}(4)$-gauge invariant and higher-spin invariant world-line correlators related to topological open-string correlators with consistent "factorization" properties manifesting themselves in higher-spin gauge-field equations.

In what follows we shall discuss the world-line correlators, the higher-spin structures and the topological closed string underlying the generalized Chan-Paton factor, and then turn in Section 4 to the open-string reformulation of the singleton.

3.6 $\mathfrak{sp}(4)$-INVARIANT WORLD-LINE CORRELATORS

Here we shall apply the phase-space approach to singleton quantum mechanics. The steps leading from (3.152) to (3.160) via (3.153) are heuristic in nature, motivated by physical considerations, and guided by algebraic structures lifted from higher-spin gauge theory.

The symmetric correlators of the free singleton theory based on the $\mathfrak{sp}(2)$-gauged action $S[Y,\Lambda] = \frac{1}{4} \int Y^{Ai} DY_{Ai}$ are given by

$$\langle \mathcal{O}_f \rangle_y = \int_{Y_i^A(\pm\infty) = y_i^A} [\mathcal{D}Y\mathcal{D}\Lambda] \ f(Y_i^A(\tau)) \ e^{iS[Y,\Lambda]} . \qquad (3.142)$$

Barring details of the BRST treatment, the gauge fixing amounts to setting the gauge field Λ_{ij} equal to a fixed value, which we shall take to be $\Lambda_{ij} = 0$, and imposing its classical equation of motion, that is $Tr[K_{ij}\mathcal{O}] = 0$. This is tantamount to decoupling ideal elements belonging to

$$\mathcal{I}[\mathfrak{sp}(2)] = \left\{ K_{ij} \star X^{ij} \in \mathcal{P}[\mathcal{Z}] \right\} , \qquad (3.143)$$

where X^{ij} denote arbitrary triplets, and $\mathcal{P}[\mathcal{Z}]$ is the space of $\mathfrak{sp}(2)$-invariant functions

$$\mathcal{P}[\mathcal{Z}] = \{ f \in \mathcal{W}[\mathcal{Z}] : [K_{ij}, f]_\star = 0 \} . \qquad (3.144)$$

This is solved by taking the observables to be elements of the weakly $\mathfrak{sp}(2)$-projected space

$$\mathcal{P}_0[\mathcal{Z}] = \frac{\mathcal{P}[\mathcal{Z}]}{\mathcal{I}[\mathfrak{sp}(2)]} , \qquad (3.145)$$

and inserting a projector – the phase-space analog of an ordinary space-time propagator – into the trace of the ungauged theory. The resulting gauged traces $Tr_\pm$ are given by

$$Tr_\pm f = tr_\pm[f \star M] , \qquad (3.146)$$

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where $M = M(K^2)$ is the *singular phase-space projector* defined by [9, 29, 11] (see also [4, 5, 6, 7])

$$K_{ij} \star M = 0 , \qquad M(0) = 1 , \qquad (3.147)$$

and $tr_{\pm}$ are the $(\pm)$-graded traces of $\mathcal{W}[\mathcal{Z}]$ defined by

$$tr_+ f = \int_{\mathcal{Z}} \frac{d^{2(D+1)}y}{(2\pi)^{D+1}} \langle \mathcal{O}_f \rangle_y = \int_{\mathcal{Z}} \frac{d^{2(D+1)}y}{(2\pi)^{D+1}} f(y) , \qquad (3.148)$$

$$tr_- f = \langle \mathcal{O}_f \rangle_0 = f(0) . \qquad (3.149)$$

One may view Tr_- and Tr_+ as singular-geometry counterparts of the bulk-to-bulk and boundary-to-boundary Green functions in anti-de Sitter spacetime, and the traces obey

$$tr_{\pm}[f(y) \star g(y)] = tr_{\pm}[g(\pm y) \star f(y)] . \qquad (3.150)$$

Drawing on the properties of the spinor-oscillator realization of the five-dimensional higher-spin gauge theory [4, 5], it has been proposed [11] that there exists a *normalizable phase-space projector* $\Delta = \Delta(K^2)$ that is a distribution living on the hypercone obeying

$$K_{ij} \star \Delta = 0 , \qquad \Delta \star \Delta = \Delta , \qquad \Delta \star M = M . \qquad (3.151)$$

The functions $M(K^2)$ and $\Delta(K^2)$ are phase-space counterparts of state-space projectors built from non-normalizable and normalizable singleton states, respectively.

An external massless particle is described by a π-invariant symmetrized two-singleton state

$$|\Psi\rangle_{12} = \left[|\alpha\rangle_1 \otimes |\beta\rangle_2 + |\widetilde{\alpha}\rangle_1 \otimes |\widetilde{\beta}\rangle_2 \right]_{\text{symm}} , \qquad (3.152)$$

where $|\alpha\rangle$ and $|\beta\rangle$ denote $\mathfrak{sp}(2)$-invariant singleton states. As drawn in Fig. 4, this state joins the internal worldline by

  i) reversal of the orientation of one of the incoming singletons;

 ii) amputation of external propagators.

Using (3.45), the operation *(i)* yields the *un-amputated vertex operator*

$$\mathcal{V} = \frac{1}{2} \left[|\alpha\rangle \otimes \langle\widetilde{\beta}| + |\beta\rangle \otimes \langle\widetilde{\alpha}| + |\widetilde{\alpha}\rangle \otimes \langle\beta| + |\widetilde{\beta}\rangle \otimes \langle\alpha| \right] , \qquad (3.153)$$

with definite transformation properties under the discrete maps π and τ given in (3.44) and (3.45). This vertex corresponds to a phase-space function,

$$\mathcal{V} = \Phi \star \kappa , \qquad (3.154)$$

where Φ belongs to the *strongly projected twisted-adjoint representation* [11]

$$T[\mathfrak{ho}_0(D-1,2)] \equiv \left\{ K_{ij} \star \Phi = \Phi \star K_{ij} = 0 , \quad \pi\tau(\Phi) = \pi(\Phi^{\dagger}) = \Phi \right\} , \quad (3.155)$$
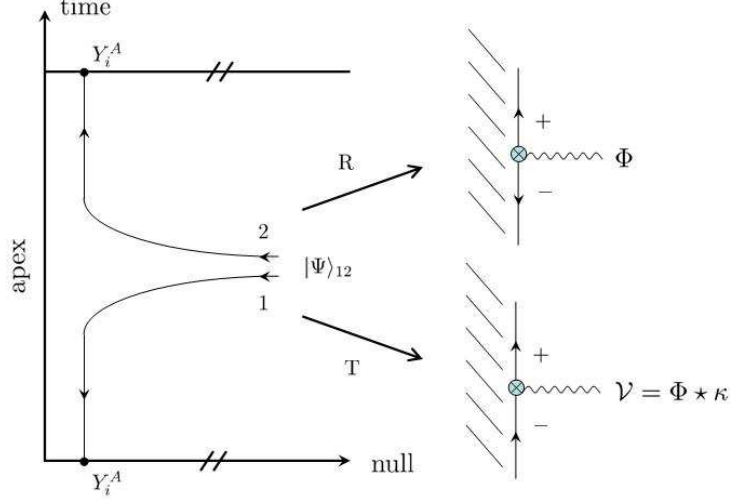
Figure 4: *Amputation and Role of Intertwiner*: An external two-singleton composite $|\Psi\rangle_{12} = |\alpha\rangle_1|\beta\rangle_2 + |\widetilde{\alpha}\rangle_1|\widetilde{\beta}\rangle_2$, is mapped to a twisted-adjoint vertex operator $\Phi = |\alpha\rangle\langle\beta| + |\widetilde{\alpha}\rangle\langle\widetilde{\beta}|$ inserted into an R-ordered disc correlator, or, equivalently, to a massless vertex operator $\mathcal{V} = \Phi \star \kappa = |\alpha\rangle\langle\widetilde{\beta}| + |\widetilde{\alpha}\rangle\langle\beta|$ inserted into a T-ordered boundary correlator. The arrows indicate the orientations of the corresponding worldlines, and the amputation amounts to replacing $\Phi = C \star M$ by $C \star \Delta$ where M and Δ are the singular and normalizable $\mathfrak{sp}(2)$ projectors.

of the higher-spin algebra $\mathfrak{ho}_0(D-1,2)$ to be defined below (see eq. (3.169)). Here π and τ acting according to (3.46) and (3.47), and κ is an oscillator implementation of the involution π given in (3.44) and (3.46), that is

$$\pi(f(y)) = \kappa \star f(y) \star \kappa , \tag{3.156}$$

$$\kappa|\alpha\rangle = |\widetilde{\alpha}\rangle , \qquad \langle\alpha|\kappa = \langle\widetilde{\alpha}| . \tag{3.157}$$

The element κ can be realized on the internal worldline as $\kappa = \exp_\star(i\pi v_A v_B w^A \star \bar{w}^B)$, where the oscillators are defined in (3.57) and (3.58) and $\exp_\star A \equiv 1 + A + \frac{1}{2}A \star A + \cdots$. This element is, however, not Weyl-ordered. As we shall see in Section 4.3, this subtlety is resolved in the two-dimensional theory, which admits a well-defined Weyl-ordered implementation of κ.

The strongly projected elements Φ can be represented as either normalizable or non-normalizable phase-space functions,

$$\Phi = \begin{cases} C \star M & \text{non-normalizable} \\ C \star \Delta & \text{normalizable} \end{cases} , \tag{3.158}$$

where C belongs to the *weakly projected twisted-adjoint representation* defined by [9]

$$T_0[\mathfrak{ho}_0(D-1,2)] = \left\{ C \in \mathcal{P}_0[\mathcal{Z}] : \quad \pi\tau(C) = \pi(C^\dagger) = C \right\} . \tag{3.159}$$

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Assuming that the amputation *(ii)* leads to normalizable states, we define the *amputated vertex operator*

$$V = C \star \Delta \star \kappa . \qquad C \in T_0[\mathfrak{ho}_0(D-1,2)] . \tag{3.160}$$

Thus, given N external massless states in a fixed cyclic order, the resulting singleton correlator reads

$$\mathcal{A}_N^{(\pm)} = Tr_\pm [V_1 \star \cdots \star V_N] = tr_\pm [C_1 \star \kappa \cdots \star C_N \star \kappa \star M] . \tag{3.161}$$

For even N, say $N = 2n$, the insertions of κ cancel, and the correlator can be expressed as a trace of Weyl-ordered operators,

$$\mathcal{A}_{2n}^{(\pm)} = tr_\pm [C_1 \star \pi(C_2) \star \cdots \star \pi(C_{2n}) \star M] . \tag{3.162}$$

Turning to the $\mathfrak{sp}(4)$-gauge transformations, the hyper-cone reflections (3.112) – which generate an anomalous transformation of the external state (3.152) – leave the twisted-adjoint field invariant, *i.e.*

$$\rho(C) = C , \tag{3.163}$$

essentially due to the fact that the phase factor cancels in

$$\rho(|\alpha\rangle\langle\beta|) = (-1)^{\epsilon_0}|\alpha\rangle\langle\beta|[(-1)^{\epsilon_0}]^\dagger = |\alpha\rangle\langle\beta| . \tag{3.164}$$

Moreover, infinitesimal $\mathfrak{sp}(4)$ transformations, given by

$$\delta V_r = \varepsilon_{(1)r}^{ij} K_{ij} \star V_r + \varepsilon_{(2)r}^{ij} V_r \star K_{ij} + (\varepsilon_r^{ij} - \varepsilon_r' \epsilon^{ij}) y_i^A \star \mathcal{V}_r \star y_{Aj} , \tag{3.165}$$

with local, *i.e.* r-dependent, parameters, leave the correlators invariant. To show this, it suffices to use the weaker $\mathfrak{sp}(2)$-invariance condition (3.144): one first observes that the transformations induced by parameters that are triplets under the $\mathfrak{sp}(2)$ vanish, since they can be written as commutators with K_{ij} and all other operators are $\mathfrak{sp}(2)$-invariant. The closure of $\mathfrak{sp}(4)$ then implies that the transformations induced by the singlets ε_r' must vanish as well.

More generally, the world-line correlators are invariant under local transformations

$$\delta V_r = \varepsilon_{(1)r} \star V_r \star \varepsilon_{(2)r} , \tag{3.166}$$

with parameters carrying general non-zero $\mathfrak{sp}(2)$ charges, and the element δV_r defining a descendant of the "primary" operator V_r. Roughly speaking, one may view the above transformations as a discrete analog of a continuous symmetry group generated by the bilinear $\mathfrak{sp}(2)$ current and stress-energy tensor studied in Section 3.8.

The space-time gauge symmetries, that will play an important role in the master-field equations to be discussed below, arise as rigid symmetries of the world-line correlators, *i.e.* transformations that leave the correlators invariant *and* preserve the $\mathfrak{sp}(2)$-projection conditions on the vertices. These transformations are

$$\delta V_r = [\epsilon, V_r]_\star , \qquad \delta \Phi_r = \epsilon \star \Phi_r - \Phi_r \star \pi(\epsilon) , \tag{3.167}$$

with rigid, *i.e.* *r*-independent, parameters ϵ belonging to the *off-shell bosonic higher-spin algebra*

$$\mathfrak{ho}(D-1,2) \; \equiv \; \left\{ P \in \mathcal{P}[\mathcal{Z}] \; : \quad \tau(P) \; = \; P^\dagger \; = \; -P \right\} , \tag{3.168}$$

containing as a subalgebra the vector-oscillator realization of the *minimal bosonic higher-spin algebra*

$$\mathfrak{ho}_0(D-1,2) \; = \; \left\{ P \in \mathcal{P}_0[\mathcal{Z}] \; : \quad \tau(P) \; = \; P^\dagger \; = \; -P \right\} , \tag{3.169}$$

where the ideal elements correspond to trivial transformations of the vertices.

Let us examine the algebraic properties of the adjoint and twisted-adjoint representations in more detail.

3.7 Adjoint and Twisted-Adjoint Fields

Here we high-light properties of the higher-spin algebra that arise directly from the singleton. The extent to which these suffice by themselves for constructing an interacting theory is clearly a challenging problem to address in its generality, linking the open string to be discussed in Section 4 to purely algebraic constructions based on spin-chains and Yangians [97, 98].

3.7.1 The Minimal Bosonic Higher-Spin Algebra

The linear action of M_{AB} in $\mathfrak{D}$ is not transitive, *i.e.* the singleton is not a fundamental representation of $\mathfrak{so}(D-1,2)$. The minimal *Lie-algebra* extension of $\mathfrak{so}(D-1,2)$ with this property is the bosonic higher-spin algebra [30, 9, 29, 11]

$$\mathfrak{ho}_0(D-1,2) \; = \; \left\{ Q \in \frac{\mathcal{U}(\mathfrak{so}(D-1,2))}{\mathcal{A}[\mathfrak{D}]} \; : \quad \tau(Q) \; = \; Q^\dagger \; = \; -Q \right\} , \tag{3.170}$$

where the singleton annihilator is defined by

$$\mathcal{A}[\mathfrak{D}] \; = \; \left\{ X \in \mathcal{U}(\mathfrak{so}(D-1,2)) \; : \quad X|\Psi\rangle \; = \; 0 \;, \quad \forall |\Psi\rangle \in \mathfrak{D} \right\} , \tag{3.171}$$

and τ is the involutive anti-automorphism defined in (A.3). The Lie bracket of $\mathfrak{ho}_0(D-1,2)$ is given by

$$[Q,Q'] \; = \; Q \star Q' - Q' \star Q \;, \tag{3.172}$$

where $\star$ is the product on $\mathcal{U}(\mathfrak{so}(D-1,2))$.

The algebra has the following *level decomposition*:

$$\mathfrak{ho}_0(D-1,2) \; = \; \bigoplus_{\ell=0}^{\infty} \mathfrak{L}_\ell \;, \tag{3.173}$$

where the ℓ'th level is an irreducible $\mathfrak{so}(D-1,2)$ tensor of highest weight $(2\ell+1, 2\ell+1)$, comprising the generators

$$Q_\ell \; = \; Q^{(2\ell+1,2\ell+1)}_{A_1...A_{2\ell+1},B_1...B_{2\ell+1}} M^{A_1 B_1} \cdots M^{A_{2\ell+1} B_{2\ell+1}} \;. \tag{3.174}$$

This can be shown using the triple-grading (A.5) of $\mathfrak{so}(D-1,2)$, and the decomposition of the singleton into energy levels defined in (A.17):

We first decompose Q_ℓ into irreducible $\mathfrak{so}(D-1,2)$ tensors, $Q_\ell = \sum_{i=1}^{r_\ell} Q_{\ell\,[i]}$. By hypothesis, the spaces $Q_{\ell\,[i]}|\epsilon_0\rangle$ are invariant under $\mathfrak{g}^{(-)} \oplus \mathfrak{g}^{(0)}$. On the other hand, the states in $Q_\ell|\epsilon_0\rangle$ have energy eigenvalues $\leq \epsilon_0 + 2\ell + 1$, and the lower energies can be reached from the maximal energy via repeated action with $\mathfrak{g}^{(-)}$, so that $Q_\ell|\epsilon_0\rangle$ is irreducible under $\mathfrak{g}^{(-)} \oplus \mathfrak{g}^{(0)}$. Hence, by a choice of the label i, $Q_{\ell\,[i]}|\epsilon_0\rangle = 0$ for $i > 1$. Acting on these identities with $\mathfrak{g}^{(+)}$, and using the irreducibility of $Q_{\ell\,[i]}$, yields $Q_{\ell\,[i]} \in \mathcal{A}[\mathfrak{D}]$ for $i > 1$, which shows the assertion that Q_ℓ is irreducible. As a byproduct, we have also found that the highest $\mathfrak{so}(D-1,2)$ weights of the ℓ'th level are given by $(2\ell+1, 2\ell+1)$, since this projection contains the generator $E^{2\ell+1}$, that acts non-trivially in $\mathfrak{D}$. $\square$

We note that the off-shell algebra (3.168) decomposes into levels given by irreducible $\mathfrak{sl}(D+1)$ tensors with highest weights $(2\ell+1, 2\ell+1)$ for $\ell = 0, 1, 2, \ldots$, with 0'th level given by $\mathfrak{so}(D-1,2)$.

The twisted-adjoint vertices are arranged into a massless multiplet of $\mathfrak{ho}_0(D-1,2)$.

The rule for multiplying $\mathfrak{so}(D-1,2)$ tensors imply

$$[Q_\ell, Q_{\ell'}]_\star = \sum_{\ell''=|\ell-\ell'|}^{\ell+\ell'} f_{\ell\ell'}{}^{\ell''} Q_{\ell''} , \qquad (3.175)$$

where the structure coefficients $f_{\ell\ell'}{}^{\ell''}$ are invariant $\mathfrak{so}(D-1,2)$ tensors times non-vanishing constants. It is now easy to see that $\mathfrak{ho}_0(D-1,2)$ is minimal: Suppose that $\mathfrak{L}$ is a Lie subalgebra of $\mathfrak{ho}_0(D-1,2)$ containing $\mathfrak{so}(D-1,2)$. Then $\mathfrak{L} = \bigoplus_\ell c_\ell \mathfrak{L}_\ell$ with multiplicities $c_\ell \in \{0,1\}$. Clearly, one possibility is $\mathfrak{L} = \mathfrak{so}(D-1,2)$, for $c_\ell = \delta_{\ell,0}$. Adding one extra generator, say at level $\ell > 0$, closure with level 0 requires adding all of $\mathfrak{L}_\ell$, and in turn, via (3.175), all levels up to 2ℓ, and so on.

The precise strengths of $f_{\ell\ell'}{}^{\ell''}$ can be evaluated using oscillator representations, relying on the *uniqueness of the minimal extension*:

To show uniqueness, we assume $\widetilde{\mathfrak{h}}$ to be a non-trivial extension of $\mathfrak{so}(D-1,2)$ that is minimal and admits $\mathfrak{D}$. Minimality implies that $\widetilde{\mathfrak{h}}$ does not contain any other ideals than $\mathfrak{so}(D-1,2)$, so that in particular $\widetilde{\mathfrak{h}} \cap \mathcal{A}[\mathfrak{D}] = 0$. Let us decompose the generators into irreducible $\mathfrak{so}(D-1,2)$ tensors, $\widetilde{Q}_{\tilde{l}}$, where $\tilde{l} = (\tilde{l}_1, \tilde{l}_2, \ldots)$ $(\tilde{l}_1 \geq \tilde{l}_2 \geq \cdots))$ are highest weights, and consider the set of generators with fixed $\tilde{l}_1$, say $\widetilde{Q}_{[i]}$, $i = 1, \ldots, r$. The spaces $\widetilde{Q}_{[i]}|\epsilon_0\rangle$ are invariant under $\mathfrak{g}^{(-)} \oplus \mathfrak{g}^{(0)}$, and consist of states with energies ranging from ϵ_0 up to $\epsilon_0 + \tilde{l}_1$. Thus, by the argument given above, $\widetilde{Q}_{[i]}|\epsilon_0\rangle = 0$ for $i > 1$ (by a choice of label i), that in turn implies $\widetilde{Q}_{[i]} \in \mathcal{A}[\mathfrak{D}]$ for $i > 1$, *i.e.* $\widetilde{\mathfrak{h}}$ contains at most one irrep for each $\tilde{l}_1$, that we can denote by $Q_{\tilde{l}_1}$. Since $\widetilde{Q}_{\tilde{l}_1}|\epsilon_0\rangle$ is irreducible under $\mathfrak{g}^{(-)} \oplus \mathfrak{g}^{(0)}$ it follows that $\widetilde{Q}_{\tilde{l}_1}|\epsilon_0\rangle = c_\ell Q_\ell|\epsilon_0\rangle$, where $\tilde{l}_1 = 2\ell+1$, the constants $c_\ell \in \{0,1\}$ (without loss of generality), and Q_ℓ $(\ell = 0, \frac{1}{2}, 1, \frac{3}{2})$ are the generators of the minimal associative higher-spin algebra $\mathcal{U}(\mathfrak{so}(D-1,2))/\mathcal{A}[\mathfrak{D}]$. Acting with $\mathfrak{g}^{(+)}$ yields $\widetilde{Q}_{\tilde{l}_1} - c_\ell Q_\ell \in \mathcal{A}[\mathfrak{D}]$. Hence, by minimality, $\widetilde{Q}_{\tilde{l}_1} = Q_\ell$ for integer ℓ, and $\widetilde{Q}_{\tilde{l}_1} = 0$ for half-integer ℓ, which completes the proof. $\square$

Having defined the algebra, we turn to its physical representations.

3.7.2 MULTIPLETONS

The Young projections $Y[\mathfrak{D}^{\otimes N}]$, referred to as multipletons[20] [40, 70], are irreducible $\mathfrak{ho}_0(D-1,2)$ representations. To show this it suffices to show that there does not exist any non-trivial $\mathfrak{ho}_0(D-1,2)$-invariant tensor $I_N : \mathfrak{D}^{\otimes N} \to \mathbb{C}$. By definition an invariant I_N obeys

$$I_N \sum_{\xi=1}^{N} Q(\xi) = 0 \quad \forall Q \in \mathfrak{ho}_0(D-1,2) , \tag{3.176}$$

and we need to show that this implies that $I_N = 0$.

For $N = 2$ and $D > 3$, the $\mathfrak{so}(D-1)$ invariance implies

$$I_2 = \sum_{n=0}^{\infty} c_n \Big[\langle E_n, (n)| \otimes \langle E_n, (n)| \Big]_{(0)} , \tag{3.177}$$

where c_n are constants, and the subscript (0) denotes the $\mathfrak{so}(D-1)$-singlet projection of $d(n) \otimes d(n)$, where $d(n)$ is defined in eq. (A.17). We note that the singlets belong to the symmetrized tensor product, so that already at this level it is clear that the anti-symmetric doubleton is irreducible. Taking $Q = E$, one finds that $\sum_{n=0}^{\infty} 2E_n c_n [\langle n| \otimes \langle n|]_{(0)} = 0$ implying $c_n = 0$ since all $E_n \neq 0$. Hence, $I_2 = 0$, and also the symmetrized doubleton is irreducible.

To show (3.176) for a general finite rank N, we use the basic properties of the π and τ maps given in (3.44) and (3.45), which imply that an element $Q \in \mathfrak{ho}_0(D-1,2)$ can be expanded as

$$Q = \sum_{\alpha,\beta} Q_{\alpha\beta} \Big(|\alpha\rangle\langle\beta| - |\widetilde{\alpha}\rangle\langle\widetilde{\beta}| \Big) , \qquad (Q_{\alpha\beta})^{\star} = -Q_{\beta\alpha} . \tag{3.178}$$

Thus, by various level truncations, it can be seen that $\mathfrak{ho}_0(D-1,2)$ contains $\mathfrak{u}(k)$ subalgebras (not containing $\mathfrak{so}(D-1,2)$) for arbitrarily large values of k, that rule out $\mathfrak{ho}_0(D-1,2)$-invariant tensors of finite rank. $\square$

Let us proceed by taking a closer look at the doubletons.

3.7.3 DOUBLETONS AND THE FLATO-FRONSDAL THEOREM

The fact that the singleton is the square root of infinitely many spins was originally found by Flato and Fronsdal in $D = 4$ [49], and later generalized to higher dimensions in [4, 7, 29].

A state $|n\rangle \in \mathfrak{D}(\epsilon_0, (0))^{\otimes 2}$ with energy $E = n + D - 3$, $n = 0, 1, 2, \ldots$, can be expanded as

$$|n\rangle = \sum_{s_1+s_2=n} \Psi^{(s_1,s_2)}_{q_1\ldots q_{s_1}, r_1\ldots r_{s_2}} (L^+_{q_1} \cdots L^+_{q_{s_1}})(1)(L^+_{r_1} \cdots L^+_{r_{s_2}})(2)|\Omega\rangle , \tag{3.179}$$

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[20]The term "doubleton" also refers to the singletons of $\mathfrak{su}(2,2) \simeq \mathfrak{so}(4,2)$, which arise most naturally from the $\mathfrak{su}(2,2)$-spinor oscillator splitting into $(2,1) \oplus (1,2)$ of $\mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Historically, Flato and Fronsdal referred to $\mathfrak{D}(1,\frac{1}{2})$ and $\mathfrak{D}(\frac{1}{2},0)$ as "singletons", presumably referring to their single-oscillator realization. They also named them "$\mathfrak{Di}$" and "$\mathfrak{Rac}$", to honor $\mathfrak{Di} \otimes \mathfrak{Rac} = \Psi_{\text{Dirac}}+$ higher-spin friends.

where $|\Omega\rangle = |\epsilon_0\rangle(1) \otimes |\epsilon_0\rangle(2)$, and the tensors $\Psi^{(s_1,s_2)}$ belong to $(s_1) \otimes (s_2)$ of the diagonal $\mathfrak{so}(D-1)$,

$$\Psi^{(s_1,s_2)}_{ppq_3...q_{s_1},r_1...r_{s_2}} = 0 = \Psi^{(s_1,s_2)}_{q_1...q_{s_1},ppr_3...r_{s_2}} \ . \tag{3.180}$$

The lowest weight condition

$$L_r^-|n\rangle = (L_r^-(1) + L_r^-(2))|n\rangle = 0 \ , \tag{3.181}$$

amounts to

$$c_n \Psi^{(n,0)}_{pq_1...q_{n-1}} + c_1 \Psi^{(n-1,1)}_{q_1...q_{n-1},p} = 0 \ , \tag{3.182}$$

$$c_{n-1} \Psi^{(n-1,1)}_{pq_1...q_{n-2},r_1} + c_2 \Psi^{(n-2,2)}_{q_1...q_{n-2},pr_1} = 0 \ , \tag{3.183}$$

$$\vdots$$

$$c_1 \Psi^{(1,n-1)}_{p,r_1\cdots r_{n-1}} + c_n \Psi^{(0,n)}_{pr_1...r_{n-1}} = 0 \ , \tag{3.184}$$

where $c_k = 2k(k + \epsilon_0 - 1)$. Eq. (3.182) forces all $\mathfrak{so}(D-1)$ irreps in $\Psi^{(s-1,1)}$ to vanish except (n), which is set equal to a constant times $\Psi^{(n,0)}$. Eq. (3.183) eliminates all irreps in $\Psi^{(n-2,1)}$ in a similar fashion, and so on. Thus, $|n\rangle$ contains precisely one lowest-weight state, given by

$$|2\epsilon_0 + n, (n)\rangle = \Psi^{(n)}_{r_1...r_n} \sum_{k=0}^{n} \frac{(-1)^k \binom{n}{k} \binom{n+\epsilon_0-1}{k}}{\binom{k+\epsilon_0-1}{k}} (L_{r_1}^+ \cdots L_{r_k}^+)(1)(L_{r_{k+1}}^+ \cdots L_{r_n}^+)(2)|\Omega\rangle \ , \tag{3.185}$$

which belongs to the symmetric tensor product for even n and the anti-symmetric part for odd n. Hence the product of two singletons decompose into two doubletons

$$[\mathfrak{D} \otimes \mathfrak{D}]_{\text{symm}} = \bigoplus_{s=0,2,4,...} \mathfrak{D}(2\epsilon_0 + s, (s)) \ , \tag{3.186}$$

$$[\mathfrak{D} \otimes \mathfrak{D}]_{\text{anti-symm}} = \bigoplus_{s=1,3,5,...} \mathfrak{D}(2\epsilon_0 + s, (s)) \ . \tag{3.187}$$

Group-theoretically, the masslessness of the doubletons expresses itself in the presence of the singular vector (A.15), corresponding to a gauge condition on a D-dimensional symmetric rank-s tensor field, or, equivalently, a $(D-1)$-dimensional conservation law. Hence $|2\epsilon_0 + s - 1, (s-1)\rangle$ must vanish identically in the singleton-composite realization (3.185) as a consequence of $L_r^+ L_r^+ |\epsilon_0\rangle = 0$, as one can check explicitly.

The doubletons have a dual enveloping-algebra presentation, referred to as the *twisted-adjoint representation*, defined by

$$T_0[\mathfrak{ho}_0(D-1,2)] = \left\{ C \in \frac{\mathcal{U}(\mathfrak{so}(D-1,2))}{\mathcal{A}[\mathfrak{D}]} \ : \ \pi\tau(C) = \pi(C^\dagger) = C \right\} \ . \tag{3.188}$$

Its elements are equivalence classes, as opposed to the strongly projected twisted-adjoint representation (3.155) whose elements are fixed by the strong $\mathfrak{sp}(2)$ projection. The equivalence classes can be represented by elements with the following level decomposition

$$C = \sum_{\ell=-1}^{\infty} C_s \ , \qquad s = 2\ell + 2 \ , \tag{3.189}$$

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$$C_s = \sum_{k=0}^{\infty} C^{(s+k,s)}_{a_1...a_{s+k},b_1...b_s} M^{a_1 b_1} \cdots M^{a_s b_s} P^{a_{s+1}} \cdots P^{a_{s+k}} , \qquad (3.190)$$

where the Lorentz traces belong to the annihilating ideal, and the $\mathfrak{ho}_0(D-1,2)$-transformation rule reads

$$\delta_\epsilon C = \epsilon \star C - C \star \pi(\epsilon) . \qquad (3.191)$$

In the real basis obeying (3.43), (3.44) and (3.45), the expansion of C reads

$$C = \sum_{\alpha,\beta} \left(C_{\alpha\beta} |\alpha\rangle\langle\beta| + (C_{\alpha\beta})^\star |\widetilde{\alpha}\rangle\langle\widetilde{\beta}| \right) , \qquad C_{\alpha\beta} = C_{\beta\alpha} , \qquad (3.192)$$

which establishes the isomorphism between $T_0[\mathfrak{ho}_0(D-1,2)]$ and $[\mathfrak{D}^{\otimes 2}]_{\mathrm{symm}}$.

3.8   CONTINUUM LIMITS

Here we examine the continuum limit $N \to \infty$ of the discretized p-brane action (3.108), and find a topological closed string inside the sigma model on the phase space of the 7D Dirac hypercone. This model is related to those of [24, 99, 100, 101, 102, 103] although our results and interpretations are slightly different. First, in the way the limits are taken, *tensile strings do not have any a priori privileged status.* Second, the singling out of $D = 7$ depends crucially on the normal-ordering prescriptions following the usage of the standard Neveu-Schwarz vacuum $|0\rangle$, in turn obeying $E_0|0\rangle = 0$ corresponding to an AdS-covariant regularization of the zero-point energy of the discrete system which differs from the regularization used in [24, 104, 105].

3.8.1   THE 7D CLOSED SINGLETON STRING

The discretized phase-space action (3.108) can be written as

$$S = \frac{1}{2} \oint d\tau \left\{ \sum_\xi (\dot{X}^A(\xi) P_A(\xi) + \Lambda^{ij} K_{ij}(\xi)) + \sum_{\xi \neq \eta} \Lambda^{i,j}(\xi,\eta) T_{i,j}(\xi,\eta) \right\} , \qquad (3.193)$$

where $K_{ij}(\xi) = \frac{1}{2} Y_i^A(\xi) Y_{Aj}(\xi)$ and $T_{i,j}(\xi,\eta) = \frac{1}{2} Y_i^A(\xi) Y_{Aj}(\eta)$. In Section 3.4 we found that effectively the physical-state conditions arise from the $\mathfrak{sp}(2)$-blocks along the diagonal and the large gauge transformations. Thus, there are two ways of taking the continuum limit: *i)* truncate $\mathfrak{sp}(2N) \to \bigoplus_\xi \mathfrak{sp}(2)_\xi$ and send $N \to \infty$ while imposing invariance under large gauge transformations by hand; *ii)* send $N \to \infty$ in $\mathfrak{sp}(2N)$.

We shall interpret the result of *(i)* as the $\mathfrak{sp}(2)$-gauged chiral (closed) string

$$S = \frac{1}{2} \int_\Sigma d\tau d\sigma (P^A \dot{X}_A + \Lambda^{ij} K_{ij}) , \qquad (3.194)$$

where the worldsheet Σ is periodic both σ and τ, in accordance with the prescriptions in (3.138) and (3.139). The weights of $(X^A, P_A; \Lambda^{ij}; K_{ij})$ under σ-reparameterizations are given by

$$\mathrm{wt}(X^A, P_A; \Lambda^{11}, \Lambda^{12}, \Lambda^{22}; K_{11}, K_{12}, K_{22}) = (0,1;0,1,2;2,1,0) . \qquad (3.195)$$

Going to the complex coordinate $z = \sigma + i\tau$, and twisting the fields using the $\mathfrak{u}(1)$ current K_{12}, one obtains a one-parameter family of models described by the chiral action ($\lambda \in \mathbb{R}$)

$$S_\lambda = \frac{1}{2} \int_\Sigma dz d\bar{z} \phi^{Ai} \bar{D} \phi_{Ai} , \tag{3.196}$$

where the conformal weights of $(\phi^{iA}; K_{ij})$ are given by

$$\text{wt}(\phi^{1A}, \phi^{2A}; K_{11}, K_{12}, K_{22}) = (\lambda, 1 - \lambda; 2(1 - \lambda), 1, 2\lambda) . \tag{3.197}$$

The levels of the $\mathfrak{sp}(2)$ currents formed out of matter and ghost fields are given by[21]

$$k_{\mathfrak{sp}(2)}^{(\phi)} = -\frac{D+1}{2} , \qquad k_{\mathfrak{sp}(2)}^{(gh)} = 4 , \tag{3.198}$$

which implies that the $\mathfrak{sp}(2)$ BRST operator is nilpotent in

$$D_{\text{crit}} = 7 . \tag{3.199}$$

Indeed, the central charge of the twisted systems, given by

$$c_{\text{tot}} = c_{(\phi)} + c_{(gh)} , \qquad c_{(\phi)} = (12\lambda^2 - 1)(D+1) , \qquad c_{(gh)} = -2(48\lambda^2 + 3) , \tag{3.200}$$

add up to a total λ-independent total central charge only for $D = 7$, where it is given by the critical value

$$c_{\text{crit}} = -14 . \tag{3.201}$$

The critical system gives rise to a BRST-invariant scalar singleton with conformal weight

$$h_{\text{crit}}(\mathfrak{D}) = -1 . \tag{3.202}$$

For example, at the manifestly $\mathfrak{sp}(2)$-invariant point $\lambda = 1/2$ the matter fields are $2(D+1)$ symplectic bosons with weight $1/2$, and the extended scalar singleton (3.53) arises in the R-sector where the zero-modes of ϕ^{Ai} act on an oscillator module created by a spin field with weight [22]

$$\lambda = \frac{1}{2} : \quad h(\mathfrak{D}) = -\frac{D+1}{8} . \tag{3.203}$$

The critical $\mathfrak{sp}(2)$-invariant system of matter and ghosts can equivalently be presented as an $\widehat{\mathfrak{so}}(6, 2)_{k_{\text{crit}}}$ WZW model with critical level identified with the free-field value, that is[23]

$$k_{\text{crit}} = -2 . \tag{3.204}$$

Indeed, the Sugawara charge for $\widehat{\mathfrak{so}}(D - 1, 2)_k$, given by

$$c_{\text{sug}} = \frac{k \dim \mathfrak{so}(D-1, 2)}{k + g} , \qquad g = D - 1 , \tag{3.205}$$

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[21] In this section we use the conventions of [106]. For a recent, related work on the equivalence between free field theories and topological WZW models, see [107].

[22] Bosonization yields the integer as well as half-integer Fock spaces (3.61) and (3.62).

[23] In $D = 7$, the stress tensor built from the total $\mathfrak{sp}(2)$ current is trivial, while the pure matter current with $k_{\mathfrak{sp}(2)}^{(\phi)} = -1 = -g(\mathfrak{sp}(2))$ yields a contracted Virasoro algebra [23, 108].

coincides with c_{crit} for the critical values of k and D, where also the weight of the primary singleton field, given by

$$h_{\mathrm{sug}}(\mathfrak{D}) \;=\; \frac{C_2[\mathfrak{so}(D-1,2)|\mathfrak{D}]}{2(k+g)} \;, \tag{3.206}$$

agrees with the free-field value.

The BRST-invariant sector of the above critical system is, however, too large, and needs to be projected further by gauging the maximal compact subalgebra $(\widehat{\mathfrak{so}}(6) \oplus \widehat{\mathfrak{so}}(2))_{-2}$ [109, 110]. Remarkably, in the critical dimension[24]

$$c_{\mathrm{sug}}[\widehat{\mathfrak{so}}(6)_{-2}] \;=\; -15 \;, \qquad c_{\mathrm{sug}}[\widehat{\mathfrak{so}}(2)_{-2}] \;=\; 1 \;, \tag{3.207}$$

so that the GKO coset construction has vanishing central charge,

$$c_{\mathrm{gko}} \;=\; -14 - (-15 + 1) \;=\; 0 \;. \tag{3.208}$$

Moreover, in the critical dimension

$$h_{\mathrm{sug}}[\widehat{\mathfrak{so}}(6)_{-2}|\mathfrak{D}] \;=\; 0 \;, \qquad h_{\mathrm{sug}}[\widehat{\mathfrak{so}}(2)_{-2}|\mathfrak{D}] \;=\; -1 \;, \tag{3.209}$$

resulting in gauged primary singletons with vanishing conformal weight

$$h_{\mathrm{gko}}(\mathfrak{D}) \;=\; -1 - (0 - 1) \;=\; 0 \;. \tag{3.210}$$

We conjecture that the corresponding spin field has a local operator product algebra so that its normal-ordered products generate a chiral ring $\mathcal{S}_{\mathrm{gko}}$ consisting group-theoretically of symmetrized singletons,

$$\mathcal{S}_{\mathrm{gko}} \;\simeq\; \bigoplus_{N=1}^{\infty} [\mathfrak{D}^{\otimes N}]_{\mathrm{symm}} \;, \tag{3.211}$$

to be identified as the generalized Chan-Paton factor. We expect the corresponding generalized operators (3.141) to be free of anomalies under the remaining large $\mathfrak{sp}(2\infty)$-gauge transformations since the 7D singleton obeys (3.137).

In $D = 7$ both vector and spinor oscillators consist of 8 complex components. The spinor oscillators give rise to the higher-spin algebra $\mathfrak{hs}(8^*)$, defined as [7]

$$\mathfrak{hs}(8*) \;=\; \{\, Q \in \frac{\mathcal{P}[\mathcal{Z}_s]}{I[\mathfrak{su}(2)]} \;:\; Q^\dagger \;=\; \tau(Q) \;=\; -Q \,\} \;, \tag{3.212}$$

where $\mathcal{Z}_s$ denotes the non-commutative spinor space and $\mathfrak{su}(2)$ is generated by spinor-oscillator bilinears. One can show that $\mathfrak{hs}(8^*)$ is minimal and admits the scalar singleton, so that by uniqueness

$$\mathfrak{hs}(8^*) \;\simeq\; \mathfrak{ho}_0(6,2) \;. \tag{3.213}$$

Hence the 7D singleton string has a dual free-field formulation based on an $\mathfrak{su}(2)$-gauged sigma model in spinor space.

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[24]In $D = 7$, the central charge vanishes also for the coset $\widehat{\mathfrak{so}}(6,2)_{-2}/\widehat{\mathfrak{so}}(6,1)_{-2}$, whose relation to the 7-dimensional scalar found in Section 3.1.4 would be interesting to spell out in more detail.

The simple tensile deformation studied in Section 3.3 is membrany in nature, and it is tantalizing to speculate about a natural embedding of the closed singleton string into String Theory would be via a discretization of the tensile supermembrane in $\text{AdS}_7 \times S^4$. However, adding free $USp(4)$ fermions to the world-sheet theory modifies the central charges.

Clearly, the above analysis leaves several issues open, in particular the details of the free-field construction of the vertex operators (3.141) and the trace operation. We plan to return to this in a more conclusive report [111].

The limit *(i)* leads to a critical system admitting a natural further projection down to the physical topological string. This raises the question whether the projection can be derived from the original $\mathfrak{sp}(2N)$ gauging by taking the limit *(ii)*.

The off-diagonal $\mathfrak{sp}(2N)$ generators with $|\xi - \eta| = n$ may be interpreted as discretizations of $\mathfrak{so}(D-1,2)$-invariant bilinears in ϕ^{Ai} containing n derivatives. At $\lambda = 1/2$, these form a higher-spin current algebra, that we denote by

$$\mathcal{W}_{\mathfrak{sp}(2)+\infty} \equiv \bigoplus_{s=1,3,5,\dots} \mathcal{W}_{ij}^{(s)} \oplus \bigoplus_{s=2,4,\dots} \mathcal{W}^{(s)} , \qquad (3.214)$$

where the currents with odd and even conformal weight s are $\mathfrak{sp}(2)$ triplets and singlets, respectively, with $\mathcal{W}_{ij}^{(1)}$ and $\mathcal{W}^{(2)}$ being the $\mathfrak{sp}(2)$ current and the free-field stress tensor, respectively. The algebra is reducible, and contains a number of subalgebras: $\mathcal{W}_{ij}^{(1)} \oplus \mathcal{W}^{(2)}$; $\mathcal{W}_\infty^{(2)}$ consisting of all even-weight currents; $\mathcal{W}_{1+\infty}$ consisting of $\mathcal{W}_\infty^{(2)}$ plus the odd-weight currents $\mathcal{W}_{12}^{(s)}$. Gauging $\mathcal{W}_{\mathfrak{sp}(2)+\infty}$, the resulting ghost contribution to the $\widehat{\mathfrak{sp}}(2)$ and Virasoro central charges become

$$k_{\mathfrak{sp}(2)}^{(gh_{\mathfrak{sp}(2)+\infty})} = \sum_{s=1,3,\dots} 1 = 0 , \qquad (3.215)$$

$$c_{\text{vir}}^{(gh_{\mathfrak{sp}(2)+\infty})} = \left(3 \sum_{s=1,3,\dots} + \sum_{s=2,4,\dots} \right) (-2)(6s^2 - 6s + 1) = 2 , \qquad (3.216)$$

that cannot be cancelled simultaneously against the matter contributions, $k_{\mathfrak{sp}(2)}^{(\phi)} = -(D+1)/8$ and $c_{\text{vir}}^{(\phi)} = -(D+1)$. Hence the continuum limit *(ii)* does not appear to lead to an interesting model.

One might instead consider gauging $\mathcal{W}_{1+\infty}$, which yields separately vanishing central charges in the matter and ghost sectors in $D = -1$. However, as we shall see, this type of critical gauging does make sense in supersymmetric setups where the central charges cancel separately in the matter sector.

3.8.2 Remarks on Twistor Superstrings

The tensionless limits of discretized standard Green-Schwarz actions will yield bosonic vector oscillators and fermionic spinor oscillators. The vectors can be converted to bosonic spinor oscillators in $D \leq 7$ as shown in [29] and demonstrated in Appendix B in the case $D = 7$, which suggests the existence of analogous dualizations of the fermionic spinors

resulting in super-oscillator analogs of (3.108) in turn leading in the continuum limit to chiral sigma models containing topological closed-string realizations of generalized super-singleton Chan-Paton factors.

It is interesting that under the vector-spinor dualities, the $\mathfrak{sp}(2)$ is converted into $\mathfrak{su}(2)$ and $\mathfrak{u}(1)$ in $D = 7$ and $D = 5$, respectively, in turn leading to master-field equations involving phase-space projectors, while the 4D spinors have trivial internal gauge group. On the other hand, in all spinor-oscillator realizations the lowest-weight state $|\epsilon_0, (0)\rangle$ is given by the Fock-space vacuum and hence has a manifestly normalizable analytical wave-function in spinor space.

An alternative approach is to view the chiral phase-space strings as fundamental. Remarkably, in this spirit, the $\mathfrak{psu}(2, 2|4)$-covariant free-field model [103] can be viewed as a critical model based on gauging the $\mathcal{W}_{1+\infty}$-extension of the free-field stress tensor and the $\mathfrak{u}(1)_Z$-current of the $\widehat{\mathfrak{su}}(2, 2|4)_0$ current algebra[25], corresponding to an $N \to \infty$ limit of a discrete $U(N)$-gauged sigma model [113]. Essentially this is due to the fact that the central charges vanish separately in the matter and ghost sectors for $(1 + \infty)$-gauging. Moreover, the central charges vanish also in the compact subalgebra $(\widehat{\mathfrak{su}}(2|2) \oplus \widehat{\mathfrak{su}}(2|2))_0$, whose gauging gives rise to a critical GKO model providing the framework for constructing the generalized Chan-Paton factor.

The above construction contains in itself no dynamics other than linearized on-shell conditions inherited from the singletons. In the case of the single singleton truncation, to be considered in the next section, the dynamics follows from the open-string reformulation of phase-space quantization, in which a crucial role is played by bi-local operators describing interacting two-singleton composites arising either as external states or internal states resulting from factorization[26]. The analogous treatment of the complete phase-space, or spinor, closed strings presumably requires an extension based on open topological membranes refined with multi-local operators.

4   THE OPEN SINGLETON STRING

In the phase-space approach, the dynamical equations are the conditions on deformations of the $\star$-product algebra preserving the gauge symmetries. Indeed, this is how the internal part of the Vasiliev equations arises from the phase-space quantization of a single singleton. One may think of the full master fields as analogs of exactly marginal operators on tensile closed-string worldsheet (or other physical deformations of tensile p-branes), and the Vasiliev equations as a classically consistent truncation of the string-field theory based on the chiral sigma model mentioned in Section 3.8.

The deformations of the singleton worldline are parameterized by $\mathfrak{sp}(2)$-projected twisted-adjoint master fields built from (X^A, P^A), which one may view as analogs of the Virasoro-projected vertices on the first Regge trajectory built from $(X^\mu, \dot{X}^\mu)$. The latter arise

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[25]We note that the $\mathcal{W}_{1+\infty}$-gauging is critical in itself without any need for internal "matter", unlike the model based on gauging the $U(1)$ and Virasoro currents, which requires an internal sector with $c_{\mathrm{vir}} = 24$ [100, 112].

[26]Dynamical phase-space Lagrangians have been proposed in [48] although their relation to the Vasiliev equations remains to be worked out.

as a sub-sector of the open-string phase space $(X^\mu, \dot{X}^\mu, \ddot{X}^\mu, \ldots)$ with non-local world-line Green function $\langle X^\mu(\tau) X^\nu(\tau') \rangle \sim i\eta^{\mu\nu} \alpha' \log|\tau - \tau'|$, while the former arise from a topological action with local Green function $\langle X^A(\tau) P^B(\tau') \rangle \sim i\eta^{AB}(\theta(\tau - \tau') - \theta(\tau' - \tau))$ that extends á la Cattaneo-Felder into a topological open string on a worldsheet Σ as we shall discuss in Sections 4.1 and 4.2.

Thus, by analogy, the *interactions giving rise to non-linear deformations require a notion of breaking up the worldline and inserting a composite realized as a topological open-string excitation* (see Fig. 4). As we shall discuss in Sections 4.3 and 4.4, this can be implemented by generalizing the local boundary condition in (3.138) to 2-*punctured boundary conditions* corresponding to adding branch-points $p \in \partial\Sigma$ where the embedding fields behave inside correlators as

$$Y^{Ai}(u, \bar{u}) \;\sim\; y^{Ai} + \frac{1}{2\pi i} y'^{Ai} \log \frac{u}{\bar{u}} + \cdots \; , \tag{4.1}$$

where u is a local coordinate vanishing at p, the omitted terms either vanish at $\partial\Sigma$ or are q-exact, and q is the shift-symmetry BRST operator. Such branch-points are generated by insertions of *bi-local operators* with a non-trivial dependence on the q-closed zero-mode y^{Ai} as well as the q-non-closed shift-mode y'^{Ai}. Indeed, the above geometric structures combined with ordinary canonical quantization yield Vasiliev's bi-local algebraic structures – the oscillator algebra (1.9), the intertwiner (1.11) and the operator d' – in turn facilitating the construction of the observable (1.15) giving rise to the internal part (1.12) and (1.13) of the weakly projected Vasiliev equations, which we shall analyze in Section 4.5.

The phase-space approach converts the standard notions of scattering amplitudes and an effective action into that of a BRST-cohomological master-field equation in non-commutative phase space $\mathcal{Z}$, which one may also think of as a non-linear phase-space counterpart of the singleton Schrödinger equation. In this formalism, the *true* space-time geometry arises as a result of Vasiliev's unfolding procedure, whose main features are high-lighted in Section 4.7.

4.1   Two-dimensional Action and $\mathfrak{sp}(2)$-Gauging

A phase space with non-degenerate symplectic structure $\Omega = d\omega$ admits Kontsevich's covariant $\star$-product, in turn identifiable with the algebra of boundary operators in the two-dimensional Cattaneo-Felder-Kontsevich model [54, 55] based on the Poisson sigma model $S = \frac{1}{2} \int_\Sigma (dY^M + \frac{1}{2} \Omega^{MN} \eta_N) \wedge \eta_M$ on a disc Σ with $\eta_M|_{\partial\Sigma} = 0$, which is a parent of the point-particle action $S = \frac{1}{2} \oint_{\partial\Sigma} \omega$.

The index contraction in the kinetic $dY^M \eta_M$-term is background independent, so that standard perturbative quantization methods give rise to a manifestly *background covariant* $\star$-product. Hence, in the case of the singleton, the subsequent master-field equations can be derived without the need to single out any specific component fields, such as metric or Lorentz connection, and this applies as well to the tentative open-membrane reformulations mentioned above.

The two-dimensional parent action of the phase-space action (3.2) is given by

$$S \;=\; \frac{1}{2} \int_\Sigma \left(DY^M \wedge \eta_M + \frac{1}{2} \eta^M \wedge \eta_M + \xi^{MN} F_{MN} \right) \; , \tag{4.2}$$

57

where $\partial\Sigma$ is the singleton worldline; η_M and $\xi^{MN} \equiv \eta^{AB}\xi^{ij}$ are Lagrange multipliers; $F^{MN} \equiv d\Lambda^{MN} + \Lambda^{MP} \wedge \Lambda_P{}^N$ with $\Lambda^{MN} = \eta^{AB}\Lambda^{ij}$; and the symplectic index $M \equiv Ai$ is raised and lowered using

$$\Omega_{MN} = -\epsilon_{ij}\eta_{AB} . \tag{4.3}$$

The stationary configurations obey

$$DY^{Ai} + \eta^{Ai} = 0 , \quad D\eta^{Ai} = 0 , \quad D\xi^{ij} + \eta^{A(i}Y_A^{j)} = 0 , \quad F^{ij} = 0 , \tag{4.4}$$

where $D\xi^{ij} = d\xi^{ij} + 2\Lambda^{k(i}\xi_k{}^{j)}$, and the boundary conditions

$$\oint_{\partial\Sigma} \delta Y^{Ai}\eta_{Ai} = 0 , \qquad \oint_{\partial\Sigma} \xi^{ij}\delta\Lambda_{ij} = 0 . \tag{4.5}$$

A unique configuration is singled out by imposing Dirichlet conditions on the one-forms,

$$\eta_{Ai}| = 0 , \qquad \Lambda_{ij}| = \text{fixed} . \tag{4.6}$$

Let us derive the relation between (4.2) and (3.2). From (4.4) it follows that

$$\xi_{ij} - K_{ij} = U_i{}^r U_j{}^s k_{rs} , \tag{4.7}$$

where $dk_{rs} = 0$ and U_{ir} is a coset element defined by $DU_{ir} \equiv dU_{ir} + \Lambda_i{}^j U_{jr} = 0$ and $U_{ir}|_p = \epsilon_{ir}$, with $p \in \Sigma$ being a given fixed point. Moreover, from (4.4) and (4.6) it follows that $D\xi_{ij}| = 0$, which implies that

$$\xi_{ij}| = U_i{}^r|U_j{}^s|k'_{rs} , \tag{4.8}$$

where k'_{rs} is another constant. The boundary field equations are therefore equivalent to the equations of motion of (3.2) with V_{ij} identified as

$$V_{ij} = U_i{}^r|U_j{}^s|(k_{rs} - k'_{rs}) . \tag{4.9}$$

Thus, from the two-dimensional point of view the singleton limit corresponds to taking $k_{rs} = k'_{rs}$, i.e.

$$K_{ij}| = 0 , \tag{4.10}$$

which we shall assume henceforth.

4.2 BRST FORMULATION AND TRUNCATION OF TRIPLETS

The systematic gauge-fixing procedure is provided by the BV formulation, for which we use the conventions and notations collected in Appendix C.

The BRST transformations leaving the field equations (4.4) invariant, are

$$\delta\eta^M = DC^M + C^{MN}\eta_N , \qquad \delta\Lambda^{MN} = -DC^{MN} , \tag{4.11}$$

$$\delta Y^M = -C^M + C^{MN}Y_N , \qquad \delta\xi^{MN} = 2C^{K(M}\xi_K{}^{N)} - Y^{(M}C^{N)} , \tag{4.12}$$

where the vector indices are suppressed, and C^{MN} and C^M are ghost fields obeying the boundary conditions

$$DC^M| = 0, \qquad DC^{MN}| = 0. \tag{4.13}$$

Demanding that δ^2 vanishes on-shell fixes

$$\delta C^M = C^{MN}C_N, \qquad \delta C^{MN} = C^{MP}C^N{}_P. \tag{4.14}$$

The resulting BRST transformations leave the classical action invariant

$$\delta S = 0. \tag{4.15}$$

The anti-field dependence of the classical BV action is expanded as

$$\mathcal{S} = \sum_n S_n + \mathcal{O}(\hbar), \qquad S_0 = S, \qquad S_1 = \sum_{\phi \in \aleph} \int_\Sigma \phi^+ \wedge \delta\phi, \tag{4.16}$$

with $\aleph = \{Y^M, \eta_M, \xi_{MN}, \Lambda^{MN}; C_M, C^{MN}; B^M, B_{MN}; \lambda^M, \lambda_{MN}\}$, where the B and λ fields, which are required to write gauge-fixing terms, are assigned the BRST transformations

$$\delta B^M = -\lambda^M, \qquad \delta\lambda^M = 0, \qquad \delta B^{MN} = \lambda^{MN}, \qquad \delta\lambda^{MN} = 0. \tag{4.17}$$

The BV-bracket can then be expanded as $(\mathcal{S}, \mathcal{S}) = (S_1, S_1) + 2(S_0, S_2) + \cdots$, where

$$(S_1, S_1) = 2\int_\Sigma \sum_{\phi \in \aleph} \phi^+ \wedge \delta^2\phi, \tag{4.18}$$

vanishes identically, i.e. $\delta\delta = 0$ off-shell. Furthermore, the action of the BV Laplacian Δ on S_1 yields terms proportional to $C^M{}_M = 0$, implying $\Delta S_1 = 0$. The BV master action is thus given by

$$\mathcal{S} = S + \sum_{\phi \in \aleph} \int_\Sigma \phi^\star \wedge \delta\phi, \tag{4.19}$$

with $\delta\phi$ given by (4.12), (4.14) and (4.17).

Using the gauge-fixing fermion

$$\Psi = \int_\Sigma \left(\eta_M \wedge \star DB^M - \Lambda^{MN} \wedge \star dB_{MN}\right), \tag{4.20}$$

where the Hodge $\star$ is defined using an auxiliary Euclidean metric, the anti-fields become

$$B_M^+ = -D \star \eta_M, \qquad \eta^{+M} = \star DB^M, \tag{4.21}$$

$$\Lambda_{MN}^+ = -\star(dB_{MN} - \eta_{(M}B_{N)}), \qquad B_{MN}^+ = d \star \Lambda_{MN}, \tag{4.22}$$

resulting in the following gauge-fixed action

$$S_{\text{gf}} = \frac{1}{2}\int_\Sigma \left\{ DY^I \wedge \eta_I + \frac{1}{2}\eta^M \wedge \eta_M + \xi_{MN}F^{MN} + \star DB^M \wedge (DC_M + C_M{}^N \eta_N) \right.$$

$$\left. + \star(dB^{MN} - \eta^M B^N) \wedge DC_{MN} + \lambda^M D \star \eta_M + \lambda^{MN} d \star \Lambda_{MN} \right\}. \tag{4.23}$$

The gauge-fixed classical field equations read

$$D\eta^M \;=\; 0\;, \qquad D\star\eta^M \;=\; 0\;, \tag{4.24}$$

$$F^{MN} \;=\; 0\;, \qquad d\star\Lambda^{MN} \;=\; 0\;, \tag{4.25}$$

$$D\star DB^M \;=\; 0\;, \qquad D\star DC^M \;=\; 0\;, \tag{4.26}$$

$$D\star dB^{MN} \;=\; 0\;, \qquad d\star DC^{MN} \;=\; 0\;, \tag{4.27}$$

$$DY^M + \eta^M + \star D\lambda^M + \star D(B^N C_N{}^M) \;=\; 0\;, \tag{4.28}$$

$$
\begin{aligned}
D\xi_{MN} + \eta_M Y_N - \star\eta_M\lambda_N + \star DB_M C_N - B_M \star DC_N + (M \leftrightarrow N) \\
= \; \star d\lambda_{MN} - 2\star dB_M{}^P C_{PN} + \star\eta_M B^P C_{NP} + (M \leftrightarrow N)
\end{aligned}
\qquad . \tag{4.29}
$$

Taking further curls yields simpler harmonic equations, of which it is worth noting $D\star d\lambda_{MN} + 2\star dB_{(M}{}^P \wedge DC_{N)P} = 0$.

The full BRST current reads

$$
\begin{aligned}
\star J \;=\;& -C^M\eta_M - Y^M DC_M + \lambda^M \star DC_M \\
& + C^{MN} \star (d\lambda_{MN} + B_M DC_N) - \star DC^{MN}\lambda_{MN} - \star dB_{MN} C^{MP} C_P{}^N \;. \tag{4.30}
\end{aligned}
$$

The BRST charge is given by integrals of $\star J$ along open contours L with endpoints at the boundary of Σ, *viz.*

$$Q \;=\; \int_L \star J\;, \qquad L \subset \Sigma\;, \qquad \partial L \subset \partial\Sigma\;. \tag{4.31}$$

The charge is conserved if $\star J| = 0$, which can be achieved (with some loss of generality) by imposing the homogenous Dirichlet conditions

$$\eta_M| \;=\; 0\;, \qquad DC^M| \;=\; 0\;, \qquad C_{MN}| \;=\; 0\;, \tag{4.32}$$

$$dB^M| \;=\; 0\;, \qquad dB_{MN}| \;=\; 0\;, \qquad \lambda^M| \;=\; 0\;, \qquad \lambda_{MN}| \;=\; 0\;, \tag{4.33}$$

where we note that the condition on C_{MN} sharpens that required by the BV master equation.

The ghost field C^M contains a zero-mode obeying

$$DC_{(0)}^M \;=\; 0\;, \qquad F_{MN} \;=\; 0\;, \tag{4.34}$$

which drops out of the gauge-fixed action. This mode must be deleted from the path-integral measure, which we define schematically as

$$\int_{\Lambda^{MN}|=\text{fixed}} \mathcal{D}\Lambda_{MN} \int_{DC^M|=0} \mathcal{D}C^M \delta(C_{(0)}^M) \int \mathcal{D}\xi^{MN}\cdots\;, \tag{4.35}$$

where the integration over ξ^{MN} ensures $F_{MN} = 0$. The ghost zero-mode corresponds to a physical zero-mode in Y^M, as can be seen from the BRST transformations given in (4.12). Moreover, the non-zero modes in Y^M are either paired with non-zero modes in C^M, in which case they are unphysical, or unpaired, in which case they in fact are BRST exact (see eq. (4.98)).

The $\mathfrak{so}(D-1,2)$ current

$$J_{AB} \;=\; Y^i_{[A} \star \eta_{B]i} + \lambda^i_{[A}\eta_{B]i} + B^i_{[A}(DC_{B]i} + C_i{}^j\eta_{B]j}) - DB^i_{[A}C_{B]i} \;, \tag{4.36}$$

in general has an anomaly, given by

$$\star J_{AB}| \;=\; \star DB^i_{[A}C_{B]i}| \;. \tag{4.37}$$

However, for certain deformations of the theory, such as those giving rise to the Vasiliev equations, it is possible to redefine the Lorentz generators, so that the resulting master-field equations are manifestly Lorentz invariant.

To simplify the gauge-fixed Lagrangian it is convenient to specify the Dirichlet condition (4.6) to the homogeneous condition

$$\Lambda^{MN}| \;=\; 0 \;. \tag{4.38}$$

At the level of field equations, the remaining $\mathfrak{sp}(2)$ triplets (recall that $\Lambda^{MN} = \eta^{AB}\Lambda^{ij}$ *idem* B^{MN} and C^{MN}) are then given by

$$\Lambda_{MN} \;=\; 0 \;, \qquad C_{MN} \;=\; 0 \;, \qquad dB_{MN} \;=\; 0 \;, \tag{4.39}$$

and

$$d\xi_{MN} \;=\; -\eta_{(M}Y_{N)} + \star\eta_{(M}\lambda_{N)} - \star dB_{(M}C_{N)} + B_{(M} \star dC_{N)} \;, \tag{4.40}$$

where the last equation is part of the $\mathfrak{sp}(2)$-invariance condition discussed in Section 4.1.

At the quantum level, and considering correlators of operators independent of the triplets, the above equations constitute a quasi-consistent truncation to the free action

$$s_{\text{gf}} \;=\; \frac{1}{2}\int_\Sigma \left(dY^M \wedge \eta_M + \frac{1}{2}\eta^M \wedge \eta_M + \lambda^M d \star \eta_M + \star dB^M \wedge dC_M \right) \;, \tag{4.41}$$

the boundary conditions $\eta_M| = 0$, $\lambda^M| = 0$, $dC^M| = 0$ and $dB^m| = 0$; the $\mathfrak{sp}(2)$ constraint following from (4.40) and (4.10); and, the shift-symmetry BRST charge

$$q \;=\; \int_L \star j \;, \qquad \star j = -C^M \eta_M - Y^M dC_M + \lambda^M \star dC_M \;, \tag{4.42}$$

generating the transformations

$$\delta_q Y^M \;=\; -C^M \;, \qquad \delta_q B^M \;=\; -\lambda^M \;. \tag{4.43}$$

Before turning to the canonical quantization, we shall discuss in more detail the boundary conditions and ordering prescriptions for boundary and bulk correlators.

4.3   On Open-String Vertex Operators

In this section we give a heuristic discussion of boundary correlators of bi-local operators, *i.e.* correlators of operators that depend on both end points of the string. The two main

proposals are that these correlators contain the $\star$ product (1.9), and that radial and time-ordered boundary correlators are related by $\star$-multiplication by the intertwining operator κ defined in (1.11).

The quantization of the model gives rise to N-*punctured correlators* of the form

$$\langle \mathcal{O} \rangle_{\{y(\xi)\}_{\xi=1}^N} \;=\; \langle T_\pm[\mathcal{O}_{\partial\Sigma}]R[\mathcal{O}_\Sigma]\rangle_{\{y(\xi)\}_{\xi=1}^N} \; , \tag{4.44}$$

where $\mathcal{O}_{\partial\Sigma}$ and $\mathcal{O}_\Sigma$ are operators inserted on $\partial\Sigma$ and Σ, respectively; $T_\pm$ denote path ordering with respect to the two orientations of $\partial\Sigma$; R denotes ordering in terms of increasing radius of concentric circles defined using an auxiliary Euclidean metric; and $y(\xi)$ are boundary conditions at points $p_\xi \in \partial\Sigma$, $\xi = 1, \ldots, N$, that we shall consider in the cases of $N = 1$ and $N = 2$.

The path orderings are related via the outer anti-automorphism τ corresponding to reversal of the orientation of $\partial\Sigma$, defined by

$$\left\langle \tau\Big(T_+[\mathcal{O}_{\partial\Sigma}]R[\mathcal{O}_\Sigma]\Big)\right\rangle \;=\; \langle T_-[\tau\mathcal{O}_{\partial\Sigma}]R[\tau\mathcal{O}_\Sigma]\rangle \; , \tag{4.45}$$

which act on the zero-mode in the embedding field as in (3.47). Moreover, letting R_p denote the ordering of the radial evolution emanating from a point $p \in \partial\Sigma$, we shall assume that $\mathcal{O}_\Sigma$ is built from operators whose mutual products are local in the sense of analytical continuation, so that

$$R_p[\mathcal{O}_\Sigma] \;=\; R[\mathcal{O}_\Sigma] \; , \tag{4.46}$$

is independent of p.

More general correlators, *e.g.* the variations of a local correlator, may require a specific choice of p and a prescription for combining the T and R orderings, which we shall refer to as an intertwiner. To formulate this algebraically, we consider the "amputation" of an external massless string state, drawn in Fig. 4, which shows that the insertions of an operator $\mathcal{O}(p)$ into the radial and path-ordered parts of $T_+[\mathcal{O}_{\partial\Sigma}]R[\mathcal{O}_\Sigma]$, respectively, are related as

$$\langle T_+ [\mathcal{O}_{\partial\Sigma}] R_p [\mathcal{O}_\Sigma \mathcal{O}(p)]\rangle \;=\; \langle T_+ [\mathcal{O}_{\partial\Sigma}(\mathcal{O}\mathcal{O}_\kappa)(p)] R [\mathcal{O}_\Sigma]\rangle \; , \tag{4.47}$$

where $\mathcal{O}_\kappa$ generates the inner automorphism corresponding to reversal of the orientation of $\partial\Sigma$, as drawn in Fig. 3, that is, $\mathcal{O}_\kappa$ implements the world-sheet extension of the automorphism π defined in (3.44) and (3.46).

Turning to the boundary data, the case $N = 1$ is implemented as the symmetric boundary condition

$$\lim_{\tau\to\pm\infty} \left\langle \mathcal{O}Y_i^A(\tau)\right\rangle_y \;=\; y_i^A \langle \mathcal{O}\rangle_y \; . \tag{4.48}$$

As a result, $Y^{Ai}|$ is given by a zero-mode identified as y_i^A plus q-exact non-zero modes, so that symmetric boundary correlators of q-invariant *local* operators $\mathcal{O}_f$, where f are functions on $\mathcal{Z}$, are equivalent to the algebra $\mathcal{W}[\mathcal{Z}]$ based on the Moyal $\star$-product (3.18) [115].

The insertion of more general operators $\mathcal{O}(p)$ requires a specification of the behavior of the embedding fields at p in the form of the 2-*punctured boundary condition*[27]

$$\lim_{\substack{\tau \to \tau(p)^\pm \\ \sigma \to 0}} \left\langle R_p \left[\mathcal{O} Y_i^A(u, \bar{u}) \right] \right\rangle_{y,y'} = (y_i^A + \frac{\arg u|_\pm}{\pi} y_i'^A) \left\langle \mathcal{O} \right\rangle_{y,y'} , \tag{4.49}$$

modulo q-exact terms, and where $u = \tau + i\sigma$ is a local coordinate vanishing at p. The determination of the phase-factor in (4.49) is chosen as

$$\arg u| = \frac{\pi}{2} \left(1 - \epsilon(\tau - \tau(p)) \right) . \tag{4.50}$$

Thus one may identify

$$y^M , \qquad \widetilde{z}^M \equiv y^M + y'^M , \tag{4.51}$$

with the phase-space coordinates of T_+ and T_--ordered portions of the boundary to the right and left of p. The T_--ordered coordinate $\widetilde{z}^M$ is mapped to a T_+-ordered counterpart z^M by the anti-automorphism τ acting as in (3.47), and we fix conventions by choosing

$$\tau(\widetilde{z}^M) = i z^M . \tag{4.52}$$

As we shall demonstrate in the next section, the resulting $\star$-product algebra for the (y, z)-oscillators is given by (1.9), and

$$Y^{Ai}| = y^{Ai} + \frac{1}{2\pi i} \log \frac{u}{\bar{u}} y'^{Ai} + \{q, \beta^{Ai}\} , \tag{4.53}$$

where the shift-mode y'^{Ai} is paired with a non-zero mode c'^{Ai} in C^{Ai} and hence unphysical. Thus, q-closed local operators $\mathcal{O}_f$ are independent of the shift-mode, and 2-punctured boundary correlators of such operators are equivalent to $\mathcal{W}[\mathcal{Z}]$.

Non-trivial deformations of the q-cohomology are generated by operators that are *covariantly q-closed*, i.e. closed up to covariantisations that drop out of the trace. The relevant operators are constructed from functions $\widehat{f}$ on $\mathcal{Z} \times \mathcal{Z}$, depending on two insertion points, *viz.* the *bi-local operators*

$$\mathcal{O}_{\widehat{f}} = \widehat{f}(Y^{Ai}(p_1), Y^{Ai}(p_2)) , \qquad p_{1,2} \in \Sigma . \tag{4.54}$$

We shall assume that in $R[\mathcal{O}_\Sigma \widehat{f}(Y^{Ai}(p_1), \widetilde{Y}^{Ai}(p_2))]$ with $p_{1,2} \in \partial\Sigma$, the boundary coordinates $Y^{Ai}(p_1)$ and $\widetilde{Y}^{Ai}(p_2)$ evolve under T_+ and T_- order, respectively. Hence, assuming that q-exact contributions cancel,

$$\left\langle T_+[\mathcal{O}_{\partial\Sigma}] R[\mathcal{O}_\Sigma \mathcal{O}_{\widehat{f}}|] \right\rangle_{y,y'} = \left\langle T_+[\mathcal{O}_{\partial\Sigma}] R[\mathcal{O}_\Sigma] \right\rangle_{y,y'} \star \widehat{f}(y, \widetilde{z}) , \tag{4.55}$$

and analogously

$$\left\langle T_+[\mathcal{O}_{\partial\Sigma} \mathcal{O}_{\widehat{f}}|] R[\mathcal{O}_\Sigma] \right\rangle_{y,y'} = \left\langle T_+[\mathcal{O}_{\partial\Sigma}] R[\mathcal{O}_\Sigma] \right\rangle_{y,y'} \star \widehat{f}(y, z) . \tag{4.56}$$

---

[27]There is also a branch-point at infinity, with asymptotic behavior determined by y and y'. Alternatively, (4.49) can be imposed as in (4.44) with p_1 and p_2 chosen as two points on the two regular branches of $\partial\Sigma$, respectively, in which case $y(p_1) = y$ and $y(p_2) = y + y'$.

Under R-ordering, the worldlines emanating from $\mathcal{O}_{\widehat{f}}$ connect to T_+-ordered worldlines to the left and T_--ordered dittos to the right. Transporting $\mathcal{O}_{\widehat{f}}$ from the R-ordered sector to the T_+-ordered sector requires a reversal of the orientation of the worldlines emanating on the right side. Thus, in (4.47) the role of $\mathcal{O}_\kappa$ is to exchange the orientation of the worldlines in accordance with the automorphism π drawn in Fig. 3. Thus, for $p_{1,2} \in \partial\Sigma$ we take

$$\widehat{f}(Y^{ai}(p_1), Y^i(p_1); \widetilde{Y}^{ai}(p_2), \widetilde{Y}^i(p_2))\, \mathcal{O}_\kappa \;\; = \;\; :\, \widehat{f}(Y^{ai}(p_1), \widetilde{Y}^i(p_2); \widetilde{Y}^{ai}(p_2), Y^i(p_1))\mathcal{O}_\kappa : \; . \tag{4.57}$$

Alternatively, applying (4.47) to $\mathcal{O}(p) = \mathcal{O}_{L_+}\mathcal{O}_{L_-}$, where $L_\pm$ denote the portions of $\partial\Sigma$ on which R_p is equivalent to $T_\pm$,

$$T_+[\mathcal{O}_{\partial\Sigma}]R[\mathcal{O}_\Sigma \mathcal{O}(p)] \;\; = \;\; T_+[\mathcal{O}_{\partial\Sigma}\left(T_+[\mathcal{O}_{L_+}]T_-[\mathcal{O}_{L_-}]\right)\mathcal{O}_\kappa]R\,[\mathcal{O}_\Sigma] \; , \tag{4.58}$$

which yields a well-defined gluing of worldlines provided

$$T_-[\mathcal{O}_{L_-}]\mathcal{O}_\kappa \;\; = \;\; \mathcal{O}_\kappa T_+[\mathcal{O}_{L_-}] \; . \tag{4.59}$$

Hence, in accordance with (4.56), $\mathcal{O}_\kappa$ can be implemented taking κ to be a bi-local phase-space function with the property[28]

$$\widehat{f}(y^{ai}, y^i; z^{ai}, z^i) \star \kappa \;\; = \;\; \kappa \widehat{f}(y^{ai}, iz^i; z^{ai}, -iy^i) \; , \tag{4.60}$$

$$\kappa \star \widehat{f}(y^{ai}, y^i; z^{ai}, z^i) \;\; = \;\; \kappa \widehat{f}(y^{ai}, -iz^i; z^{ai}, iy^i) \; , \tag{4.61}$$

$$\kappa \star \widehat{f}(y^{ai}, y^i; z^{ai}, z^i) \star \kappa \;\; = \;\; \widehat{f}(y^{ai}, -y^i; z^{ai}, -z^i) \; , \tag{4.62}$$

which we identify with the action of the intertwiner in (1.11).

With these preliminary considerations in mind, we turn to the canonical quantization with the primary aim of deriving the $\star$-product algebra and q-transformations of the (y, z)-oscillators.

4.4 $\star$-PRODUCTS AND q-TRANSFORMATIONS

In this section we analyze the oscillator algebra and BRST transformations of the free field theory defined by the truncated Lagrangian (4.41). To begin, we choose the auxiliary metric to be the flat metric on the upper half-plane

$$\Sigma \;\; = \;\; \{u = \tau + i\sigma : \sigma > 0\} \; , \qquad ds^2 \;\; = \;\; |du|^2 \; , \qquad \star du \;\; = \;\; -idu \; , \tag{4.63}$$

and consider the radial evolution around the point $p \in \partial\Sigma$ with $u(p) = 0$, governed by the field equations

$$d \star \eta^M \;\; = \;\; d\eta^M \;\; = \;\; 0 \; , \quad dY^M + \eta^M + \star d\lambda^M \;\; = \;\; 0 \; , \tag{4.64}$$

subject to the boundary conditions

$$\eta^M| \;\; = \;\; 0 \; , \quad \lambda^M| \;\; = \;\; 0 \; , \tag{4.65}$$

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[28]The classical multiplication by κ may follow from boundary terms in turn required as counter-terms in order for the variational principle (4.5) to hold also at the branch points.

$$d \star dB^M \;=\; d \star dC^M \;=\; 0\,, \qquad dB^M| \;=\; dC^M| \;=\; 0\,, \tag{4.66}$$

imposed on

$$\partial\Sigma \setminus \{p\} \;=\; L_+ \cup L_-\,, \qquad L_\pm \;=\; \{\sigma = 0\,, \pm\tau > 0\}\,. \tag{4.67}$$

The fields are also subject to the Euclidean-signature reality conditions

$$(Y^M(u,\bar{u}))^\dagger \;=\; Y^M(u,\bar{u})\,, \quad (\lambda^M(u,\bar{u}))^\dagger \;=\; \lambda^M(u,\bar{u})\,, \tag{4.68}$$

$$(B^M(u,\bar{u}))^\dagger \;=\; B^M(u,\bar{u})\,, \qquad (C^M(u,\bar{u}))^\dagger \;=\; C^M(u,\bar{u})\,. \tag{4.69}$$

The resulting mode expansions read

$$Y^M \;=\; y^M + \frac{1}{2\pi i} y'^M \log \frac{u}{\bar{u}} + i \sum_{n\neq 0} \frac{1}{n}(y_n^M u^{-n} - \bar{y}_n^M \bar{u}^{-n})\,, \tag{4.70}$$

$$\lambda^M \;=\; i \sum_{n\neq 0} \frac{1}{n}\mathrm{Im}y_n^M(u^{-n} - \bar{u}^{-n})\,, \tag{4.71}$$

$$B^M \;=\; b^M + \frac{1}{2\pi i} b'^M \log \frac{u}{\bar{u}} + i \sum_{n\neq 0} \frac{1}{n} b_n^M(u^{-n} - \bar{u}^{-n})\,, \tag{4.72}$$

$$C^M \;=\; c^M + \frac{1}{2\pi i} c'^M \log \frac{u}{\bar{u}} + i \sum_{n\neq 0} \frac{1}{n} c_n^M(u^{-n} - \bar{u}^{-n})\,, \tag{4.73}$$

where $\bar{y}_n^M = (y_n^M)^\dagger$ and the remaining operators are real, and we have also given the results for the ghosts for later reference.

To determine the (y,z)-oscillator algebra it suffices to compute the radial-ordered two-point functions subject to the homogeneous boundary condition

$$\lim_{|u|\to\infty} \left\langle R\left[Y^M(u,\bar{u})\mathcal{O}\right]\right\rangle_0 \;=\; 0\,. \tag{4.74}$$

The ηY and $\eta\lambda$ contractions are the inverses of the kinetic terms in

$$s_{\mathrm{kin}}[Y,\eta,\lambda] \;=\; \frac{1}{2}\int_\Sigma \left(dY^M \wedge \eta_M + \lambda^M d\star\eta_M\right)\,. \tag{4.75}$$

Using conventions in which $\int f(u,\bar{u})d\star d\frac{1}{4\pi}\log|u|^2 = \int f(u,\bar{u})dd\frac{1}{4\pi i}\log\frac{u}{\bar{u}} = f(0,0)$, one finds

$$\left\langle R\left[\eta_M(1)Y^N(2)\right]\right\rangle_0 = d_1\Psi(1,2)\delta_M^N\,, \qquad \left\langle R\left[\eta_M(1)\lambda^N(2)\right]\right\rangle_0 = -id_1\Phi(1,2)\delta_M^N\,, \tag{4.76}$$

$$\Psi(1,2) \;=\; \frac{1}{2\pi}\log\frac{u_1-u_2}{\bar{u}_1-\bar{u}_2}\frac{\bar{u}_1-u_2}{u_1-\bar{u}_2}\,, \qquad \Phi(1,2) \;=\; \frac{i}{2\pi}\log\frac{|u_1-u_2|^2}{|u_1-\bar{u}_2|^2}\,. \tag{4.77}$$

The YY contractions arise via the vertex $\frac{i}{4}\int_\Sigma \eta^M \wedge \eta_M$, leading to

$$\left\langle R\left[Y^M(1)Y^N(2)\right]\right\rangle_0 \;=\; 2\times\frac{i}{4}\int_\Sigma \left\langle Y^M(1)\eta^P(u,\bar{u})\right\rangle_0 \left\langle \eta_P(u,\bar{u})Y^N(2)\right\rangle_0\,, \tag{4.78}$$

that can be evaluated by dissecting Σ into three parts: two discs, D_k $(k=1,2)$, with evanescent radii centered on u_k; and the remaining part of the upper half-plane, Σ', with

branch cuts L_k from $\bar{u}_k$ to u_k. The contributions from D_k vanish, while that from Σ' can be rewritten using Stokes' theorem as

$$\int_{\Sigma'} d\Psi(1) \wedge d\Psi(2) = \oint_{\partial\Sigma'} \Psi(1) d\Psi(2) - \int_{\Sigma'} \Psi(1) d^2\Psi(2) , \qquad (4.79)$$

where $d = du\partial_u + d\bar{u}\partial_{\bar{u}}$, $\Psi(k) \equiv \Psi(u, \bar{u}; k)$ and the last term vanishes identically, since $\Psi(u, \bar{u}; 2)$ is regular in Σ'. In the first term

$$\partial\Sigma' = \{\text{Re } u = 0\} \cup \bigcup_k (-\partial D_k) \cup L_{k,+} \cup (-L_{k,-}) , \qquad (4.80)$$

where $L_{k,\pm}$ denote contours drawn on each side of the two branch cuts. The contributions from $\{\text{Re } u = 0\} \cup (-\partial D_1) \cup L_{2,+} \cup (-L_{2,-})$ vanish, and

$$\oint_{-\partial D_2} \Psi(1) d\Psi(2) = -2i\Psi(2,1) , \qquad (4.81)$$

$$\int_{L_{1,+} \cup (-L_{1,-})} \Psi(1) d\Psi(2) = 2i \int_q^1 d\Psi(2) = 2i\Psi(1,2) , \qquad (4.82)$$

where it has been used that $\Psi(1)|_+ - \Psi(1)|_- = 2i$ and $\Psi(u|, \bar{u}|; 2) = 0$.

Hence, we have arrived at

$$\langle R\left[Y^M(1)Y^N(2)\right]\rangle_0 = -\Omega^{MN}\{\Psi(1,2) - \Psi(2,1)\} = \frac{1}{\pi}\Omega^{MN}\log\frac{u_1 - \bar{u}_2}{\bar{u}_1 - u_2} . \qquad (4.83)$$

In the limit $\text{Im } u_k \to 0$, this two-point function yields

$$\langle R\left[Y^M(\tau_1)Y^N(\tau_2)\right]\rangle_0 = -i\Omega^{MN}\epsilon(\tau_1 - \tau_2) , \qquad (4.84)$$

that in turn produces the $T_\pm$-ordered two-point functions

$$\langle T_\pm\left[Y^M(\tau_1)Y^N(\tau_2)\right]\rangle_0 = -i\Omega^{MN}\epsilon(\tau_1 - \tau_2) , \qquad \pm\tau_{1,2} > 0 , \qquad (4.85)$$

from which it follows that

$$\langle y^M \star y^N\rangle_0 = -i\Omega^{MN} , \qquad \langle \widetilde{z}^M \star \widetilde{z}^N\rangle_0 = i\Omega^{MN} . \qquad (4.86)$$

To determine their mutual contractions it suffices to consider the following holomorphic moments of the R-ordered two-point function,

$$\langle y'^M \star Y^N(v, \bar{v})\rangle_0 = \oint_{|u|>|v|} du\frac{\partial}{\partial u}\langle R\left[Y^M(u, \bar{u})Y^N(v, \bar{v})\right]\rangle_0 = 2i\Omega^{MN} , \qquad (4.87)$$

$$\langle Y^M(v, \bar{v}) \star y'^N\rangle_0 = \oint_{|u|<|v|} du\frac{\partial}{\partial u}\langle R\left[Y^M(v, \bar{v})Y^N(u, \bar{u})\right]\rangle_0 = 0 , \qquad (4.88)$$

$$\langle y'^M \star y'^N\rangle_0 = \oint_{|u|>|v|} du\oint_0 dv\frac{\partial}{\partial u}\frac{\partial}{\partial v}\langle R\left[Y^M(u, \bar{u})Y^N(v, \bar{v})\right]\rangle_0 = 0 . \qquad (4.89)$$

Setting $\text{Im } v = 0$ then gives

$$\langle y'^M \star y^N\rangle_0 = 2i\Omega^{MN} , \qquad \langle y^M \star y'^N\rangle_0 = \langle y'^M \star y'^N\rangle_0 = 0 , \qquad (4.90)$$

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which are consistent with (4.86), and we note that y'^M is a commuting element.

The resulting contraction rules read

$$\langle y^M \star y^N \rangle_0 \;=\; \langle y^M \star \widetilde{z}^N \rangle_0 \;=\; -\langle \widetilde{z}^M \star y^N \rangle_0 \;=\; -\langle \widetilde{z}^M \star \widetilde{z}^N \rangle_0 \;=\; -i\Omega^{MN} \;, \quad (4.91)$$

or equivalently, using (4.52), as

$$\langle y^M \star y^N \rangle_0 \;=\; i\,\langle y^M \star z^N \rangle_0 \;=\; -i\,\langle z^M \star y^N \rangle_0 \;=\; \langle z^M \star z^N \rangle_0 \;=\; -i\Omega^{MN} \;, \quad (4.92)$$

which we identify as the contraction rules following from Vasiliev's $\star$-product rule (1.9). In [114] this "non-commutativity" of the string end points was analyzed classically.

Since y and z arise under a common T_+-ordered evolution, it follows from (4.45) that their contractions are invariant under the anti-involution τ, and for the same reason also under $\dagger$, whose transformation properties we identify as

$$\begin{aligned}
\tau(\widehat{f}(y^M, z^M)) &\;=\; \widehat{f}(iy^M, -iz^M)\;, & \tau(\widehat{f} \star \widehat{g}) &\;=\; \tau(\widehat{g}) \star \tau(\widehat{f})\;, & (4.93)\\
(\widehat{f}(y^M, z^M))^\dagger &\;=\; \widehat{f}^\dagger(y^M, z^M)\;, & (\widehat{f} \star \widehat{g})^\dagger &\;=\; \widehat{g}^\dagger \star \widehat{f}^\dagger\;. & (4.94)
\end{aligned}$$

The linearized q-transformations read

$$[q, \mathrm{Re}\, y_n^M] \;=\; -c_n^M\;, \qquad \{q, b_n^M\} \;=\; -\mathrm{Im}\, y_n^M\;, \tag{4.95}$$

$$[q, y'^M] \;=\; -c'^M\;, \qquad [q, y^M] \;=\; 0\;, \tag{4.96}$$

$$\{q, b^M\} \;=\; \{q, b'^M\} \;=\; 0\;, \tag{4.97}$$

where the contribution from the zero-mode c^M drops out due to the δ function inserted into the measure (4.35). The resulting cohomology is thus generated by y^M and the two B^M-ghost modes b^M and b'^M. Thus, the two-singleton system in Fig. 4 does not depend on the relative distance between the two singletons[29]. Moreover, on the boundary the embedding field is given by

$$Y^M| \;=\; y^M + \frac{1}{2\pi i} y'^M \log \frac{u}{\bar{u}}| - 2 \left\{ q\,,\; \sum_{n \neq 0} \frac{b_n^M}{n} \tau^{-n} \right\}\;, \tag{4.98}$$

justifying (4.55) and (4.56).

The form of the q-transformations (4.95) and (4.97) implies that q acts as the exterior derivative (1.14) in the ghost-extended system of master fields of the form

$$\widehat{\Omega} \;=\; \widehat{\Omega}(y^M, z^M, dz^M)\;, \qquad dz^M \;=\; c'^M\;. \tag{4.99}$$

In what follows we shall, however, truncate the ghost sector, and build observables using only matter fields, using classical Grassmann-odd parameters in the q-transformations.

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[29] In case $\partial\Sigma$ has $N-1$ punctures, p_ξ, $\xi = 1, \dots, N-1$, there are $N-1$ shift-modes $y_\xi'^M$ paired with corresponding ghost modes $c_\xi'^M$, so there still remains a single physical bosonic zero-mode.

4.5 THE VASILIEV OBSERVABLES

So far we have exhibited a number of features of the two-dimensional topological field theory based on the undeformed action (4.2). This model describes not only the free propagation of topological open strings on the singleton phase space, but also their interactions. In principle, it should be possible to assemble the data contained in the BRST operator and a suitable set of fundamental string vertices into an interacting open-string field action. Here we shall instead exploit an alternative approach, whereby open-string field equations are derived by requiring that deformations of (4.2) preserve the BRST symmetry. This is analogous to marginal deformations of ordinary tensile string theory, such as those used to derive the effective field equations in the massless sector. Clearly, the drawback of this approach is the somewhat *ad hoc* introduction of the deformations, as opposed to the unified treatment offered by a proper action approach.

Let us consider the observables

$$
\widehat{Tr}_\pm \left\{ T_+ \big[\exp \oint_{p_2 \in \partial\Sigma} dY^M \widehat{A}'_M \big] \ R \big[\exp \frac{i}{2} \int_{p_2 \in \Sigma} dY^i \wedge dY_i \widehat{\Phi}' \big] \right\} \bigg|_{p_1 \in \partial\Sigma} , \qquad (4.100)
$$

with traces to be specified below, and where the operators are bi-local with $Y^{Ai}(p_1)$ identified with the zero-mode y^{Ai} in accordance with (4.55) and (4.56) (justified by (4.98)), *i.e.*

$$
\widehat{A}'_M \ = \ \widehat{A}'_M(y^N, Y^N(\tau(p_2))) \ , \qquad \widehat{\Phi}' \ = \ \widehat{\Phi}'(y^N, Y^N(u(p_2), \bar{u}(p_2))) \ . \qquad (4.101)
$$

The second argument is integrated over $\partial\Sigma$ or Σ, and we shall assume that the surface deformation is local in the sense of (4.46).

Using (4.43), the variations under shift-symmetry BRST transformations with local parameter ε^M are given by

$$
\delta_{\varepsilon q} T[e^{\oint \widehat{A}'}] \ = \ T[e^{\oint \widehat{A}'} \oint dY^M \varepsilon^N \widehat{F}'_{MN}] \ , \qquad (4.102)
$$

$$
\delta_{\varepsilon q} R[e^{\frac{i}{2} \int d^2 Y \widehat{\Phi}'}] \ = \ -iR \left[e^{\frac{i}{2} \int d^2 Y \widehat{\Phi}'} \big(\oint \varepsilon^i dY_i \widehat{\Phi}' + \frac{1}{2} \int dY^i (dY_i \varepsilon^{aj} - 2\varepsilon_i dY^{aj}) \partial_{aj} \widehat{\Phi}' \big) \right] (4.103)
$$

where $\widehat{F}'_{MN}$ are the components of the field strength

$$
\widehat{F}'_{MN} \ = \ \partial_M \widehat{A}'_N + \widehat{A}'_M \star \widehat{A}'_N - (M \leftrightarrow N) \ , \qquad (4.104)
$$

with $\partial_M = \partial/\partial Y^M(\tau)$, and $\star$ referring to the operator product on $\partial\Sigma$. The non-abelian extension arises from the expansion of $T[e^{\oint \widehat{A}'}]$ in terms of T-ordered products of integrals $\int_L \widehat{A}'$ over open intervals $L \subset \partial\Sigma$, with $\delta_{\varepsilon q} \int_L \widehat{A}'$ given by one contribution on L which yields the abelian part and one contribution on ∂L which covariantises lower orders in the expansion of the exponential.

In order to cancel the variations on $\partial\Sigma$ we use the intertwining relation (4.47) and the replacement (4.56) (again relying on (4.98)), that is

$$
\widehat{A}'_M(y^N, Y^N(\tau)) \bigg| \rightarrow \widehat{A}'_M(y^N, z^N) \ , \qquad \widehat{\Phi}'(y^M, Y^N(\tau)) \mathcal{O}_\kappa \bigg| \rightarrow \widehat{\Phi}'(y^M, z^N) \star \kappa \ , \quad (4.105)
$$

resulting in the component form[30] of the internal two-form curvature constraint (1.12). The one-form curvature constraint (1.13) then follows from integrability [9], in turn implying that the bulk variation in (4.103) can be rewritten as $[\widehat{A}'_{ai}, \widehat{\Phi}']_\pi$ and then cancelled by means of (4.47), the graded cyclicity and assuming the $\mathfrak{sp}(2)$-invariance that assures that $\widehat{\Phi}'$ is an even element.

Turning to the traces, the truncated $\mathfrak{sp}(2)$-triplet sector must be replaced by a singular projector, following essentially the same reasoning as in Section 3.6 using full $\mathfrak{sp}(2)$-generators inducing canonical rotations of all doublet indices $\widehat{A}'_{Ai}(y^{Bj}, z^{Bj})$ and $\widehat{\Phi}'(y^{Ai}, z^{Ai})$ [9]. In the oscillator formulation, these can be shown to be

$$\widehat{K}'_{ij} = K^{(tot)}_{ij} - \frac{1}{4}\{\widehat{S}'^A_i, \widehat{S}'_{Aj}\}_\star = K_{ij} - \partial^A_{(i}\widehat{A}'_{j)A} + 2\widehat{A}'^A_{(i} \star \widehat{A}'_{j)A} , \qquad (4.106)$$

where

$$\widehat{S}'_{Ai} = z_{Ai} - 2i\widehat{A}'_{Ai} . \qquad (4.107)$$

Consequently, the observables (4.100) require the deformed trace

$$\widehat{Tr}_\pm \mathcal{O} = \widehat{tr}_\pm[\mathcal{O} \star \widehat{M}] , \qquad (4.108)$$

with $\widehat{M}$ defined by (1.8), and $\widehat{tr}_\pm$ defined by

$$\widehat{tr}_+ \widehat{\mathcal{O}} = \int \frac{d^{2(D+1)}y \, d^{2(D+1)}y'}{(2\pi)^{2(D+1)}} \, \langle \widehat{\mathcal{O}} \rangle_{y,y'} , \qquad (4.109)$$

$$\widehat{tr}_- \widehat{\mathcal{O}} = \widehat{tr}_+[\widehat{\mathcal{O}} \star \rho] , \qquad (4.110)$$

with ρ an implementation of the large gauge transformation (3.40), determined by the oscillator algebra (1.9) to be

$$\rho = \exp\left(y^M z_M\right) , \qquad (4.111)$$

with the properties

$$\rho \star \widehat{f}(y, z) = \rho \widehat{f}(-iz, iy) , \qquad \widehat{f}(y, z) \star \rho = \rho \widehat{f}(iz, -iy) . \qquad (4.112)$$

Concerning the Lorentz invariance, eq. (4.37) implies that $\mathfrak{so}(D-1, 2)$-anomalies are induced by non-vanishing C^{Ai}-ghosts on the boundary corresponding to $dY^{Ai}|$ insertions in the observable (4.100). The Lorentz generators can be replaced, however, by improved full dittos generating canonical Lorentz transformations of all Lorentz indices in the master fields [9, 11],

$$\widehat{M}'_{ab} = M^{(tot)}_{ab} - \frac{1}{2}\{\widehat{S}'^i_a, \widehat{S}'_{bi}\}_\star = M_{ab} - \partial^i_{[a}\widehat{A}'_{b]i} + 2\widehat{A}'^i_{[a} \star \widehat{A}'_{b]i} , \qquad (4.113)$$

where $M^{(tot)}_{ab} = M_{ab} + \frac{1}{2}z^i_a z_{bi}$. We note that $\widehat{M}'_{ab}$ reduces to M_{ab} in the gauge where $\widehat{A}'_{ai} = 0$ and the remaining master-field components are taken to be independent of z^{ai}.

At the level of the master-field equations, the minimal bosonic model based on the adjoint and twisted-adjoint representations given in (3.168) and (3.159), respectively, results from

---

[30]Inclusion of ghosts should lead to direct contact with (1.12) via the identification (4.99).

imposing the discrete symmetries on the full master fields using π, τ and $\dagger$ given by (4.62), (4.93) and (4.94). This corresponds to unoriented open strings, leading to factorization of (4.100) using symmetrized phase-space propagators along the lines stressed in Section 3.6, and we hope to report on this elsewhere.

This concludes our analysis of the Vasiliev observables (4.100), which clearly leaves a number of important details to be worked out, as discussed in Section 1.2 below eq. (1.15).

We next turn to some remarks on the 4D spinor string, and then end our considerations by discussing some aspects of the unfolding procedure.

4.6 REMARK: 4D SPINOR STRING

The 4D bosonic spinor string is defined by the action

$$
S \;=\; \frac{1}{2} \int_{\Sigma} (dY^\alpha \wedge \eta_\alpha + \frac{1}{2}\eta^\alpha \wedge \eta_\alpha + d\bar{Y}^{\dot\alpha} \wedge \bar{\eta}_{\dot\alpha} + \frac{1}{2}\bar{\eta}^{\dot\alpha} \wedge \bar{\eta}_{\dot\alpha}) \;, \tag{4.114}
$$

where Y^α and $\bar{Y}^{\dot\alpha} = (Y^\alpha)^\dagger$ embed the worldsheet into $SL(2,\mathbb{C})$-spinor space $\mathcal{Z}$. There is no internal gauge group, the world-line correlators are invariant under the spinor-realization $\mathfrak{hs}(4) \simeq \mathfrak{ho}_0(3,2)$ [1, 116, 117], and Vasiliev's $\star$-product algebra of functions $\widehat{f}(y^\alpha, \bar{y}^{\dot\alpha}; z^\alpha, \bar{z}^{\dot\alpha})$ [2] follows from the boundary correlators of bi-local operators in a fashion completely analogous to that of the vector model.

However, unlike the vector model, the π-map, which is originally defined by its action on the higher-spin algebra induced via (A.4), can be lifted to two inequivalent maps π and $\bar\pi$ reflecting the holomorphic and anti-holomorphic coordinates of $\mathcal{Z}$, respectively, generated by

$$
\kappa \;=\; e^{iy^\alpha z_\alpha} \;, \qquad \bar\kappa \;=\; e^{-i\bar{y}^{\dot\alpha}\bar{z}_{\dot\alpha}} \;. \tag{4.115}
$$

This suggests a complexification in which κ and $\bar\kappa$ intertwine boundary correlators involving explicit holomorphic and anti-holomorphic free-field insertions, respectively. We propose that with this prescription the observable

$$
\widehat{tr}_\pm \left\{ T\big[\exp \oint_{\partial\Sigma} (dY^\alpha \widehat{A}_\alpha + \text{h.c.})\big]\ R\big[\exp \frac{i}{4}(\int_\Sigma dY^\alpha \wedge dY_\alpha b\widehat{\Phi} + \text{h.c.})\big] \right\} \;, \tag{4.116}
$$

where $\widehat{tr}_-[\mathcal{O}] = \widehat{tr}_+[\mathcal{O} \star (\kappa\bar\kappa)]$, has q-transformations proportional to the 4D spinor-oscillator Vasiliev equations, *viz.*

$$
\widehat{F} \;=\; \frac{ib}{4}dz^\alpha \wedge dz_\alpha \widehat{\Phi} \star \kappa - \text{h.c.} \;, \qquad \widehat{D}\widehat{\Phi} \;=\; 0 \;, \tag{4.117}
$$

with curvatures given by direct analogs of those in (1.5). The truncation to the minimal bosonic Type A and Type B models [118] for $b = 1$ and $b = i$, respectively, then follows the steps out-lined above for the vector model, with analogous definitions of τ and $\dagger$ and either π or $\bar\pi$ can be used to define the twisted-adjoint master field. The observables (4.116) are zero-form invariants on $\mathcal{M}$, and we expect the expansion of the Wilson-loop to produce the invariants examined recently in [65].

In both (4.100) and (4.116), the two-component nature of Y^i and Y^α, respectively, is crucial for the q-variation of the surface deformation to simplify, hindering a straightforward extension to spinor strings in $D > 4$. This is the basic reason that for the supersymmetric algebras [28, 6, 7]

$$\mathfrak{hs}(8|4) \supset \mathfrak{osp}(8|4) \ , \qquad \mathfrak{hs}(2,2|4) \supset \mathfrak{psu}(2,2|4) \ , \qquad \mathfrak{hs}(8^*|4) \supset \mathfrak{osp}(8^*|4) \ , \qquad (4.118)$$

only the corresponding 4D theory is known fully, based on an extension of (4.117) by a single fermionic oscillator in a vector representation of $SO(8)$ (see [28] for details). The open-string realization leads, however, to interesting modifications, such as a fermionic superpartner of z_α, and also suggests natural refinements, such as a fundamental 2-form potential.

4.7 SPACE-TIME UNFOLD AND THE DOUBLING PROPOSAL

In the doubling proposal, the background-covariant quantization in the fiber $\mathcal{Z}$ leads to weakly projected bi-local master fields $\widehat{\Phi}'$ and $\widehat{A}'$. This data is in turn unfolded á la Vasiliev into a classical geometry

$$E[\mathcal{M}; \widehat{\Phi}, \widehat{A}] \ , \qquad (4.119)$$

which is a bundle with base manifold $\mathcal{M}$, and sections given by bi-local master fields obtained by extending $\widehat{\Phi}'$, $\widehat{A}'$ and d' from rank-r differential forms on $\mathcal{Z}$ to rank-r dittos on $\mathcal{M} \times \mathcal{Z}$, viz.

$$\widehat{d} \ = \ dx^\mu \partial_\mu + dz^M \partial_M \ , \qquad (4.120)$$
$$\widehat{A} \ = \ dx^\mu \widehat{A}_\mu(x^\nu; y^N, z^N) + dz^M \widehat{A}_M(x^\nu; y^N, z^N) \ , \qquad (4.121)$$
$$\widehat{\Phi} \ = \ \widehat{\Phi}(x^\mu; y^M, z^M) \ , \qquad (4.122)$$

obeying i) identical kinematical constraints with $\tau(x^\mu) = (x^\mu)^\dagger = x^\mu$ and

$$\tau(\widehat{A}_\mu) \ = \ (\widehat{A}_\mu)^\dagger \ = \ -\widehat{A}_\mu \ , \qquad (4.123)$$

so that $\widehat{A}_\mu \in \mathfrak{ho}(D-1,2)$ defined in (3.168); and ii) the extended strongly projected Vasiliev equations (1.5) and (1.6).

Since *all curvatures tangent to $\mathcal{M}$ vanish*, the bi-local master fields can be expressed in coordinate balls $U \subset \mathcal{M}$ using a gauge function,

$$\widehat{A}_\mu \ = \ \widetilde{g}^{-1} \star \partial_\mu \widehat{g} \ , \quad \widehat{A}_M \ = \ \widetilde{g}^{-1} \star (\widehat{A}'_M + \partial_M) \star \widehat{g} \ , \quad \widehat{\Phi} \ = \ \widetilde{g}^{-1} \star \widehat{\Phi}' \star \pi(\widehat{g}) \ , \qquad (4.124)$$

in turn determining local master fields

$$A_\mu \ = \ \widehat{A}_\mu|_{z=0} \ , \qquad \Phi \ = \ \widehat{\Phi}|_{z=0} \ , \qquad (4.125)$$

containing the physical fields given by the scalar $\phi = \Phi|_{y=0}$ and symmetric-tensor gauge fields given by in general Φ-dependent directions in the y-expansion of A_μ [11]. Due to these field redefinitions, the metric and symmetric-tensor gauge fields are non-trivial even for simple choices of gauge function [65].

The global geometry is in general non-trivial. This state of affairs – that a given Weyl zero-form determines a geometry – is completely analogous to that in lower-spin systems involving gravity, only that in higher-spin theory the vielbein is extended by an infinite tower of higher-spin gauge fields and a scalar packed away into local master fields Φ and A_μ in turn incorporated into bi-local master fields $\widehat{\Phi}$ and $\widehat{A}$ for which the local gauge fixing assumes the simple form (4.124). In this respect, the higher-spin extension presents a tremendous simplification, facilitating a purely algebraic calculation of the metric and its higher-spin counterparts starting from $\widehat{\Phi}'$ [65].

Alternatively, determining the z-dependence in a Φ-expansion, the remaining constraints are equivalent to

$$\widehat{F}_{\mu\nu}|_{z=0} \; = 0 \; , \qquad \widehat{D}_\mu\widehat{\Phi}|_{z=0} \; = \; 0 \; , \tag{4.126}$$

constituting an integrable set of differential constraints on the local master fields. Formally, the y-expansion of (4.126) yields an infinite set of fundamental r_i-forms α^i and composite interactions $f^i(\alpha^j)$ obeying

$$d\alpha^i + f^i \; = \; 0 \; , \qquad f^j \wedge \frac{\partial f^i}{\partial \alpha^j} \; = \; 0 \; , \tag{4.127}$$

defining a non-linear cohomology, or graded homotopy Lie algebra, which can also be expressed as a nilpotent $(d + Q)$-structure acting on arbitrary composites $W(\alpha^i)$ [16]

$$(d + Q)W \; = \; 0 \; ; \qquad Q \; \equiv \; -f^i\frac{\partial}{\partial\alpha^i} \; , \tag{4.128}$$

$$d^2 W \; = \; \{d, Q\}W \; = \; Q^2 W \; = \; 0 \; . \tag{4.129}$$

By construction, the system is invariant under the gauge transformations

$$\delta\alpha^i \; = \; d\epsilon^i + \epsilon^j \wedge \frac{\partial f^i}{\partial\alpha^j} \; , \tag{4.130}$$

with $\epsilon^i \equiv 0$ if $r_i = 0$, so that locally each positive-rank form can be gauged away, which is essentially the non-linear version of the Poincaré's lemma.

Starting from $E[\mathcal{M}; \widehat{\Phi}, \widehat{A}]$ one may view the open-string field $\widehat{\Phi}' = \widehat{\Phi}|_{x=0}$ as an ultra-local holographic dual of the higher-spin gauge theory in the simply connected coordinate ball U. More generally, the homotopy invariance – which does require the vielbein to be invertible – facilitates many other holographic duals.

Referring to a D-dimensional base manifold; taking the vielbein e^a to be invertible; assuming the solution to be asymptotically Weyl flat [65]; requiring regularity in the center of spacetime; and assuming the existence of an *on-shell action* $S = \int_{\mathcal{M}} L_D$ where L_D is a suitable Q-closed D-form [16], the perturbative expansion around the anti-de Sitter metric yields holographic n-point correlation functions $\mathcal{C}_n(\widehat{X}_1, \ldots, \widehat{X}_n)$ where $\widehat{X}$ are boundary coordinates.

Assuming instead the vielbein to admit a null direction given by a globally well-defined vector field R obeying $i_R e^a = 0$, $a = 0, \ldots, D-1$, the Vasiliev equations imply flow equations along R. In particular, the perturbative expansion around Dirac's D-dimensional

hypercone defines an evolution from an initial value at $R = \infty$ to a final value at $R = 0$. For the zero-form, the constraint in (4.126) reads

$$\nabla_{(0)}\Phi + \frac{1}{2i}e_{(0)}^a\{P_a, \Phi\} + P \ = \ 0 \ , \tag{4.131}$$

where $\nabla_{(0)}$ contains the hyper-cone Lorentz connection and P is linear in the $e_{(0)}^a$ and given by an expansion in fluctuation fields starting in the second order. Hitting (4.131) with i_R yields the flow equations

$$\nabla_{(0)R}\Phi + i_R P \ = \ 0 \ . \tag{4.132}$$

Focusing on the scalar-field sector, the independent component fields are given by

$$\phi \ = \ \Phi|_{y=0} \ ; \qquad \widetilde{\phi} \ = \ (\ker e^a) \cap \phi_a \ , \quad \phi_a \ = \ i\frac{\partial^2\Phi}{\partial Y^{ai}\partial Y_i}\Big|_{y=0} \ , \tag{4.133}$$

where $e_\mu{}^a$ is viewed as a linear map from Lorentz vectors to tangent vectors, and they obey

$$\mathcal{L}_R\phi \ = \ -i_R P|_{y=0} \ , \qquad \mathcal{L}_R\widetilde{\phi} \ = \ -i_R\widetilde{P} \ . \tag{4.134}$$

Based on the $\mathfrak{ho}_0(D-1,2)$ symmetry and the properties of (4.134) – whose right-hand side vanishes in the leading order – we propose that $\mathcal{C}_n(\widehat{X}_1,\ldots,\widehat{X}_n)$ are the correlators of bilinear operators built from a singleton φ on the conifold, and that $P[\phi,\widetilde{\phi},\ldots]$ is related to the β-functional $\beta[\phi,\widetilde{\phi},\ldots]$ governing the anomalous scale dependence of renormalized finite local coupling constants.

Under this hypothesis, the scalars ϕ and $\widetilde{\phi}$ are related to the source of $\mathcal{O} = \varphi^2$ and a Lagrange multiplier used to switch on an $\mathcal{O}^2$-interaction giving rise to the scale [38]. It is interesting that in $D = 7$ the above model interpolates between a free theory in the IR with $\Delta(\mathcal{O}) = \Delta_+ = 4$ and a UV fixed point with $\Delta(\mathcal{O}) = \Delta_- = 6 - 4 = 2$, that is, $\mathcal{O}$ and the interaction $\mathcal{O}^2$ become a fundamental free field and its mass-term, respectively, serving as a form of UV "safety net".

5 CONCLUSIONS

A new angle on string quantization in curved backgrounds is provided by the doubling proposal, whereby the quantum strings live in fibers over classical space-time "unfolds". The crucial input is the singleton determining the structure group of the fiber – the classical geometry is then an output. The structure group contains a higher-spin algebra acting "diagonally" on the generalized Chan-Paton factor – and as gauge group determining the self-interactions for massless doubletons. The full structure group is some Yangian-style extension by "off-diagonal" generators, giving rise to a presently unknown massive gauge theory.

The massless sector lives on the boundary of an open string – corresponding to a single singleton worldline – and the massive sector lives on the boundary of an open membrane – corresponding to a topological closed string containing a chiral ring generated by a singleton-valued weight-0 spin field. The purely bosonic closed string has anomalies that

cancel in $D = 7$, while the analogous singleton worldline has only a global anomaly that cancels in $D = 3$ mod 4, raising a tricky question of whether the open-string extension should be anomaly free as well.

Historically, higher-spin gauge theory was developed using remarkably simple algebraic techniques, and we end by speculating over the massive theory in the same spirit. The fact that the massless gauge theory consists of total-rank 0 and 1 master fields, suggests a massive theory consisting of master fields of unlimited rank, say $\widehat{A}$ and $\widehat{B}$ with ghost numbers $1, 3, \ldots$ and $2, 4, \ldots$, respectively, subject to

$$\widehat{dA} + \widehat{A} \star \widehat{A} + \widehat{B} = 0 , \qquad \widehat{dB} + \widehat{A} \star \widehat{B} - \widehat{B} \star \widehat{A} = 0 . \tag{5.1}$$

This system becomes dynamical upon truncating $\widehat{B} \to \widehat{\Phi} \star \widehat{J}'$, where $\widehat{\Phi}$ is a zero-form and $\widehat{J}'$ a fixed intertwiner. An (associative) $\star$-product emulation of the multi-parton system can be realized using multi-flavored oscillators, with $\widehat{f}_M(\{y(\xi), z(\xi)\}_{\xi=1}^M) \star \widehat{f}_N(\{y(\xi), z(\xi)\}_{\xi=1}^N)$ given by "cluster" expansion, $\widehat{f}_M \star \widehat{f}_N = \sum_{k=1}^{\min(M,N)} \widehat{f}_{M+N-k}$, wherein k denotes the numbers of pairs of oscillators that are identified and Weyl ordered.

Acknowledgements:

A   CONVENTIONS FOR $\mathfrak{so}(D-1,2)$ REPRESENTATION THEORY

The $\mathfrak{so}(D-1,2)$ generators M_{AB} obey the commutation rules

$$[M_{AB}, M_{CD}] \;=\; i\eta_{BC}M_{AD} - i\eta_{AC}M_{BD} - i\eta_{BD}M_{AC} + i\eta_{AD}M_{BC} \;, \qquad (A.1)$$

where $\eta_{AB} = \mathrm{diag}(--+\cdots+)$ and the $\mathfrak{so}(D-1,2)$ vector index $A = 0', 0, 1, \ldots, D-1$. The decomposition into translations and Lorentz rotations is given by

$$P_a \;=\; v_a{}^A v^B M_{AB} \;, \qquad M_{ab} \;=\; v_a{}^A v_b{}^B M_{AB} \;, \qquad (A.2)$$

where $(v_a^A, v^A) \in SO(D-1,2)/SO(D-1,1)$, and $\eta_{ab} = \mathrm{diag}(-+\cdots+)$. The enveloping algebra $\mathcal{U}(\mathfrak{so}(D-1,2))$ has the outer anti-involution τ, defined by

$$\tau(M_{AB}) \;=\; -M_{AB} \;. \qquad (A.3)$$

The combined time reversal and parity operation π is the outer involution defined by

$$\pi(P_a) \;=\; -P_a \;, \quad \pi(M_{ab}) \;=\; M_{ab} \;. \qquad (A.4)$$

The real form $\mathfrak{oso}(D-1,2)$ has maximal compact subalgebra $\mathfrak{g}^{(0)} \simeq \mathfrak{so}(2) \oplus \mathfrak{so}(D-1)$. The eigenvalues of the adjoint $\mathfrak{so}(2)$ action define a triple grading

$$\mathfrak{so}(D-1,2) \;=\; \mathfrak{g}^{(-)} \oplus \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(+)} \;, \qquad [E, \mathfrak{g}^{(p)}] \;=\; p\mathfrak{g}^{(p)} \;, \qquad (A.5)$$

corresponding to the following decomposition of the generators

$$\mathfrak{so}(2)_E \;\; : \;\; E \;=\; P_0 \;=\; M_{0,0'} \;, \qquad\qquad\qquad\qquad (A.6)$$
$$\mathfrak{so}(D-1)_S \;\; : \;\; M_{rs} \;, \qquad\qquad\qquad\qquad\qquad\qquad (A.7)$$
$$\mathfrak{g}^{(\pm)} \;\; : \;\; L_r^\pm \;\equiv\; M_{0r} \mp iP_r \;=\; M_{0r} \mp iM_{r,0'} \;, \qquad (A.8)$$
$$[E, L_r^\pm] \;=\; \pm L_r^\pm \;, \quad [L_r^-, L_s^+] \;=\; 2(\delta_{rs}E + iM_{rs}) \;. \qquad (A.9)$$

where the energy operator E is the generator of time translations; the spin operators M_{rs} are the generators of spatial rotations; and $L_r^\pm$ are the generators of energy-raising and energy-lowering boosts.

The D-dimensional one-particle and one-anti-particle states form unitary irreducible representations of $\mathfrak{so}(D-1,2)$ of lowest-weight type with energy bounded from below and above by eigenvalues $E_0 > 0$ and $-E_0 < 0$, respectively. These representations are denoted by $\mathfrak{D}(E_0, S_0)$ and $\widetilde{\mathfrak{D}}(-E_0, S_0)$, where $S_0 = (s_1, \ldots, s_\nu)_{D-1}$ ($\nu = [(D-1)/2]$) is the spin, *i.e.* highest $\mathfrak{so}(D-1)$ weights, carried by the states with extremal energy. We shall let $(s_1, \ldots, s_\mu)$ denote $(s_1, \ldots, s_\mu, 0, \ldots, 0)$ for $\mu < \nu$.

The lowest-weight spaces are obtained starting from generalized Verma modules[31] $V(E_0, S_0)$ obtained boosting the ground state $|E_0, S_0\rangle$ obeying the generalized lowest-weight condition

$$L_r^- |E_0, S_0\rangle \;=\; 0 \;, \qquad (A.10)$$

---

[31] The compact subalgebra generators are suppressed in the generalized modules.

and then factoring out the maximal ideal $\mathcal{N}(E_0, S_0)$ consisting of all submodules generated from singular vectors in $V(E_0, S_0)$, *i.e.* excited states in $V(E_0, S_0)$ obeying the lowest-weight condition. In other words,

$$\mathfrak{D}(E_0, S_0) \;=\; V(E_0, S_0)/\mathcal{N}(E_0, S_0) \;, \tag{A.11}$$

where

$$V(E_0, S_0) \;=\; \bigoplus_{n=0}^{\infty} \bigoplus_{S} \big\{ |E_n, S\rangle = \underbrace{L_{r_1}^+ \cdots L_{r_n}^+ |E_0, S_0\rangle}_{S \text{ projected}} \big\} \;, \tag{A.12}$$

and, using the natural inner product induced via

$$\langle E_n, S| \;=\; \big(L_{r_1}^+ \cdots L_{r_n}^+ |E_0, S_0\rangle \big)^{\dagger} \;=\; \langle E_0, S_0| L_{r_1}^- \cdots L_{r_n}^- \;, \tag{A.13}$$

the maximal ideal can be said to consist of all states that vanish in the inner-product sense, that is

$$\mathcal{N}(E_0, S_0) \;=\; \big\{ |E_n, S\rangle \in V(E_0, S_0) : \; \langle E_n, S|E_n, S\rangle \;=\; 0 \big\} \;. \tag{A.14}$$

The non-degenerate inner-product space $\mathfrak{D}(E_0, S_0)$ is positive definite only for certain values of the lowest weights falling into three categories [119]: massive, massless and singleton-like. The latter two yield non-trivial singular vectors, notably the longitudinal modes of the massless symmetric rank-s tensor, given by

$$|s + 1 + 2\epsilon_0, (s-1)\rangle_{r_1 \ldots r_{s-1}} \;=\; L_r^+ |s + 2\epsilon_0, (s)\rangle_{r r_1 \ldots r_{s-1}} \;, \tag{A.15}$$

and the scalar-singleton mass-shell condition

$$|2 + \epsilon_0, (0)\rangle \;=\; L_r^+ L_r^+ |\epsilon_0, (0)\rangle \;. \tag{A.16}$$

The singleton comprises a single line in weight space,

$$\mathfrak{D} \;=\; \bigoplus_{n=0}^{\infty} d(n) \;, \tag{A.17}$$

where each $d(n)$ is an $\mathfrak{so}(2)_E \oplus \mathfrak{so}(D-1)_S$ irrep with energy eigenvalue $\epsilon_n = \epsilon_0 + n$ and spin (n).

One-particle and one-anti-particle spaces can be paired in a natural fashion by extending the π-map (A.4) to an involutive map acting also on states, defined by

$$\pi|E_0, S_0\rangle \;=\; |-E_0, S_0\rangle \;. \tag{A.18}$$

From $\pi L_r^- = L_r^+$, $\pi E = -E$ and $\pi M_{rs} = M_{rs}$ it then follows that

$$\widetilde{\mathfrak{D}}(-E_0, S_0) \;=\; \pi(\mathfrak{D}(E_0, S_0)) \;, \tag{A.19}$$

so that $\mathfrak{D}(E_0, S_0) \oplus \widetilde{\mathfrak{D}}(-E_0, S_0)$ becomes irreducible under $\{\mathfrak{so}(D-1,2), \pi\}$.

B VERIFICATION OF $\mathfrak{ho}_0(6,2) \simeq \mathfrak{hs}(8^*)$

Let us verify the isomorphism (3.213) by showing that $\mathfrak{hs}(8^*)$ and $\mathfrak{ho}_0(6,2)$ are isomorphic real forms of a complex algebra based on the $GL(8;\mathbb{C})$-invariant $\star$-product algebra $\mathcal{W}[\mathcal{X}]$ generated by

$$[X_I, \bar{X}^J]_\star = 2\delta_I^J , \tag{B.1}$$

where X_I and $\bar{X}^J$ are not subject to any reality condition. The complex higher-spin algebra is given by

$$\mathfrak{ho}_0(S;\mathbb{C}) = \{ Q \in \frac{\mathcal{W}[\mathcal{X}]}{I[H_S]} : \tau(Q) = -Q \} , \tag{B.2}$$

where $\tau(f(X,\bar{X})) = f(iX, i\bar{X})$, and $I[H_S]$ is the ideal generated by left and right $\star$-multiplication with the generators of $H_S \simeq A_1$ given by[32]

$$L_+ = \tfrac{1}{4}\bar{X}^I \bar{X}^J S_{IJ} , \qquad L_- = -\tfrac{1}{4}X_I X_J S^{IJ} , \qquad L_3 = \tfrac{1}{4}\bar{X}^I X_I , \tag{B.3}$$

where $S_{IJ} = S_{JI}$ and $S^{IJ}S_{KJ} = \delta_K^I$. The $GL(8;\mathbb{C})$ transformations $X \to gX$, $\bar{X} \to \bar{X}(g^{-1})^T$ and $S \to gSg^T$ can be used to change the signature S. For fixed S, the maximal finite-dimensional subalgebra of $\mathfrak{ho}_0(S;\mathbb{C})$ is $\mathfrak{so}(S;\mathbb{C})$ generated by $i\bar{X}^I \Sigma_I{}^J X_J$ where $S^{IJ}\Sigma_J{}^K + (I \leftrightarrow K) = 0$ and $\Sigma_I{}^J S_{JK} + (I \leftrightarrow K) = 0$. The algebra $\mathfrak{ho}_0(S;\mathbb{C})$ decomposes under $\mathfrak{so}(S;\mathbb{C})$ into levels with highest weights $(2\ell+1, 2\ell+1)$, $\ell = 0,1,2,\ldots$. Using the basis

$$Q_{I(\ell),J(\ell)} \equiv M_{I_1 J_1} \cdots M_{I_{2\ell+1} J_{2\ell+1}} , \tag{B.4}$$

where $M_{IJ} = iX_I \bar{X}^K S_{KJ}$ and $I(\ell) \equiv I_1 \ldots I_{2\ell+1}$, the resulting structure coefficients of $\mathfrak{ho}_0(S;\mathbb{C})$ are

$$[Q_{I(\ell),J(\ell)}, Q_{K(\ell'),L(\ell')}] = \sum_{k=0}^{2\min(\ell,\ell')} c_{I(\ell),J(\ell);K(\ell'),L(\ell')}^{M(\ell+\ell'-k),N(\ell+\ell'-k)} Q_{M(\ell+\ell'-k),N(\ell+\ell'-k)} , \tag{B.5}$$

where the coefficients are $\mathfrak{so}(S;\mathbb{C})$-invariant tensors built from products of S_{IJ} and δ_I^J.

Real forms $\mathfrak{ho}_0(S_\epsilon)$ arise from reality conditions on the oscillators,

$$\bar{X}^I = (X_J)^\dagger M_J{}^I , \qquad \det M \neq 0 , \qquad M^\star = M , \tag{B.6}$$

as

$$\mathfrak{ho}_0(S_\epsilon) = \{ Q \in \frac{\mathcal{W}[\mathcal{X}]}{I[H_\epsilon]} : \tau(Q) = Q^\dagger = -Q \} , \tag{B.7}$$

where H_ϵ is defined by (B.3) with $S \equiv S_\epsilon$ obeying

$$M(S_\epsilon)^\star M^T = -\epsilon S^{-1} , \qquad H_\epsilon = \begin{cases} \mathfrak{su}(2) & \epsilon = +1 \\ \mathfrak{sp}(2) & \epsilon = -1 \end{cases} \tag{B.8}$$

---

[32]We use conventions in which $[L_+, L_-]_\star = 2L_3$ and $[L_3, L_\pm]_\star = \pm L_\pm$. The two inequivalent real forms are $(L_3)^\dagger = L_3$ and $(L_\pm)^\dagger = \epsilon L_\mp$, giving $\mathfrak{su}(2)$ and $\mathfrak{sp}(2)$ for $\epsilon = +1$ and $\epsilon = -1$, respectively.

The algebras $\mathfrak{hs}(8^*)$ and $\mathfrak{ho}_0(6,2)$ arise from

$$\mathfrak{hs}(8^*) \quad : \quad M \;=\; i\Gamma^{08}\,, \qquad S \;=\; C\,, \qquad (\epsilon = -1)\,, \tag{B.9}$$

$$\mathfrak{ho}_0(6,2) \quad : \quad M \;=\; \eta\,, \qquad S \;=\; \eta\,, \qquad (\epsilon = +1)\,, \tag{B.10}$$

and their structure coefficients are given by (B.5) with S_{IJ} replaced by $C_{\alpha\beta}$ and η_{AB}, respectively. These are numerically the same when expressed in spinor indices α and vector indices A indices.

Finally, to convert between vector and spinor indices we use a triality-rotation matrix E_A^α obeying

$$E_A^\alpha E_B^\beta C_{\alpha\beta} \;=\; \eta_{AB}\,, \qquad \Gamma^9 E_A \;=\; E_A\,. \tag{B.11}$$

This matrix is invariant under $SO(6,2)' \otimes SO(6,2)''$ rotations, and hence the symbols

$$\Gamma^{AB}_{CD} \;=\; E_C^\alpha E_D^\delta (\Gamma^{AB})_{\gamma\delta}\,, \tag{B.12}$$

are invariant under the diagonal $SO(6,2)$. From $\Gamma^{(AB)}_{CD} = 0 = \Gamma^{AB}_{(CD)}$ it then follows that $\Gamma^{AB}_{CD} = -\frac{1}{4}\delta^{AB}_{[CD]}$, in turn implying

$$E_A^\alpha E_B^\beta M_{\alpha\beta} = M_{AB}\,, \tag{B.13}$$

which together with (B.11) implies that conversion between spinor and vector indices in the explicit commutators descending from (B.5) does not affect the normalization of the structure coefficients.

C BV Formalism

The BV expectation value is given by [120]

$$\langle \mathcal{O} \rangle_\Psi \;=\; \int d\phi\, d\phi^+ \delta\left(\phi - \frac{\partial\Psi}{\partial\phi^+}\right) \mathcal{O} \exp \frac{i}{\hbar}\mathcal{S}\,, \tag{C.1}$$

where ϕ^α and ϕ_α^+ are the fields and anti-fields, respectively; Ψ is the gauge fermion; $\mathcal{S} = \int_{\mathcal{M}} \mathcal{L}[\phi,\phi^+]$ is the BV action; and $\mathcal{O}$ are the observables. The set of fields constitutes all the integration variables, *i.e.* the classical fields, ghosts and Lagrange multipliers. The set of anti-fields constitutes the basis of the dual of the infinite-dimensional space of differential forms on field space. The fields and anti-fields are further characterized by Grassmann parity $\epsilon(\phi^\alpha) + \epsilon(\phi_\alpha^+) = 1$ mod 2; ghost number $\mathrm{gh}(\phi^\alpha) + \mathrm{gh}(\phi_\alpha^+) = -1$; differential-form degree on $\mathcal{M}$, $\deg(\phi^\alpha) + \deg(\phi_\alpha^+) = \dim\mathcal{M}$; and that they have dual boundary conditions on $\partial\mathcal{M}$. Finally, the gauge fermion $\Psi = \Psi[\phi,\phi^+;\gamma]$ is a functional with $\epsilon(\Psi) = 1$, $\mathrm{gh}(\Psi) = -1$ and $\deg(\Psi) = 0$, depending on an auxiliary metric γ on $\mathcal{M}$.

The integrand is a formally well-defined measure, *i.e.* if Ψ and Ψ' are two different gauge fermions then

$$\langle \mathcal{O} \rangle_\Psi \;=\; \langle \mathcal{O} \rangle_{\Psi'}\,, \tag{C.2}$$

provided that

$$\Delta\left(\mathcal{O} \exp \frac{i}{\hbar}\mathcal{S}\right) = 0\,, \qquad \Delta \;\equiv\; (-1)^{\epsilon^a+1}\frac{\partial_r}{\partial\phi^\alpha}\frac{\partial_r}{\partial\phi_\alpha^+}\,, \tag{C.3}$$

where ∂_r denotes derivatives taken from the right. This is an infinite-dimensional analog of the fact that $\int f$ is well-defined if $df = 0$, with $\Delta \leftrightarrow d$, $d\phi^\alpha \leftrightarrow \frac{\partial_r}{\partial\phi_\alpha^+}$, $\phi_\alpha^+ \leftrightarrow i_\alpha$, so that the dual basis of forms $(1, \phi_\alpha^+, \phi_\alpha^+ \phi_\beta^+, \dots) \leftrightarrow (V, i_\alpha V, i_\alpha i_\beta V, \dots)$, where V is the volume form. The condition (C.3) amounts to the master equations

$$(\mathcal{S}, \mathcal{S}) - 2i\hbar\Delta\mathcal{S} \;=\; 0 \;, \tag{C.4}$$

$$\mathcal{QO} \;\equiv\; (\mathcal{O}, \mathcal{S}) - i\hbar\Delta\mathcal{O} \;=\; 0 \;, \qquad \mathcal{Q}^2 \;=\; 0 \;, \tag{C.5}$$

where the anti-bracket, defined via

$$\Delta(XY) \;=\; X\Delta Y + (-1)^{\epsilon(Y)}(\Delta X)Y + (-1)^{\epsilon(Y)}(X, Y) \;, \tag{C.6}$$

is given explicitly by

$$(X, Y) \;=\; \frac{\partial_r X}{\partial\phi^\alpha}\frac{\partial_l Y}{\partial\phi_\alpha^+} - \frac{\partial_r X}{\partial\phi_\alpha^+}\frac{\partial_l Y}{\partial\phi^\alpha} \;=\; X\left(\frac{\overleftarrow{\partial}}{\partial\phi^\alpha}\frac{\overrightarrow{\partial}}{\partial\phi_\alpha^+} - \frac{\overleftarrow{\partial}}{\partial\phi_\alpha^+}\frac{\overrightarrow{\partial}}{\partial\phi^\alpha}\right)Y \;. \tag{C.7}$$

The bracket $(\cdot, \cdot)$ has a grading that can be derived by assigning the comma ghost number 1, implying $\epsilon((X, Y)) = \epsilon(X) + \epsilon(Y) + 1$, $\mathrm{gh}((X, Y)) = \mathrm{gh}(X) + \mathrm{gh}(Y) + 1$ and

$$0 \;=\; (X, Y) + (-1)^{(\epsilon(X)+1)(\epsilon(Y)+1)}(Y, X) \;, \tag{C.8}$$
$$0 \;=\; ((X, Y), Z) + (-1)^{(\epsilon(X)+1)(\epsilon(Y)+\epsilon(Z))}((Y, Z), X)$$
$$\quad + (-1)^{(\epsilon(Z)+1)(\epsilon(X)+\epsilon(Y))}((Z, X), Y) \;. \tag{C.9}$$

The master equation for $\mathcal{S}$ is solved in a double ϕ^+-expansion and $\hbar$-expansion subject to the boundary condition and regularity conditions

$$\mathcal{S}|_{\phi^+=\hbar=0} \;=\; S[\phi_{\mathrm{cl}}] \;, \qquad \det\frac{\partial_l\partial_r\mathcal{S}}{\partial\phi^\alpha\partial\phi_\beta^+} \;\neq\; 0 \;, \tag{C.10}$$

and the solution is assumed to be unique up to canonical transformations generated by $\mathcal{S}$. The $\hbar$-corrections may be thought of as Green-Schwarz anomaly cancellation terms. The formalism extends to the effective master action Γ for which (Γ, Γ) measures the true quantum anomalies. The operators are found likewise up to generalized BRST transformations $\delta\mathcal{O} = (\mathcal{S}, X) - i\hbar\Delta X$.

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